Leader Notes: Backwards and Forwards

Purpose:

In this Explore phase participants are investigating multiple ways (concretely, tabularly, graphically, symbolically, etc.) of teaching the concepts and procedures of inverse functions.

Descriptor:

Participants will explore inverses of functions concretely using patty paper to perform reflections of the graph of a relation, numerically by examining number patterns found in tables of data values, then symbolically determining the inverse of a given function and whether or not two relations or functions are inverses of each other.

Duration:

3 hours

TEKS:

a5 Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.1 **Foundations for functions.** The student uses properties and attributes of functions and applies functions to problem situations.

2A.1A The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations;

2A.1B The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

2A.2 **Foundations for functions.** The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

2A.2A The student is expected to use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

2A.4 **Algebra and geometry.** The student connects algebraic and geometric representations of functions.

2A.4A The student is expected to identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

2A.4B The student is expected to extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

2A.4C The student is expected to describe and analyze the relationship between a function and its inverse.

TAKS™ Objectives Supported:

While the Algebra 2 TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 5: Quadratic Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance: Poster-size version of **Processing Framework Model**, **Vocabulary Model** template copied onto cardstock or colored paper

Presenter Materials: blank transparency (optional – chart paper could be used) and transparency pen(s),

Per group: chart paper, markers

Per participant: graphing calculator, several sheets of patty paper, sticky notes, copies of participants' pages

Leader Notes:

Many participants will likely be familiar with procedures associated with inverses of functions. During this portion of the professional development, be sure to emphasize conceptual understanding of inverses of functions. Connect the known procedures with each other and anchor them in conceptual underpinnings.

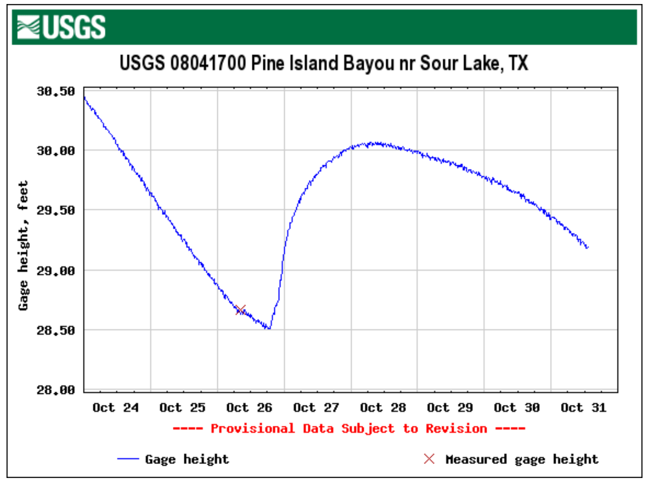
In a typical workshop setting, Parts 1, 2, and 3 can be done consecutively with discussion at the end of each part. Parts 4 and 5 can be jigsawed among groups; after the jigsaw is completed, then collectively draw generalizations about inverses of functions. Parts 6 and 7 (which deal with compositions) can serve as extensions of inverses into Precalculus notions of composition of functions.

Explore

<u>Part 1</u>: Generating an Inverse Relation

Note to Leader: At the end of Part 1, participants should be able to recognize inverse relations graphically as a reflection across the line $y = x$.

Post the scatterplot of gage height vs. time for Pine Island Bayou for participants to view. You may wish to use Transparency: Pine Island Bayou. Or, you may wish to copy the image into a PowerPoint presentation.



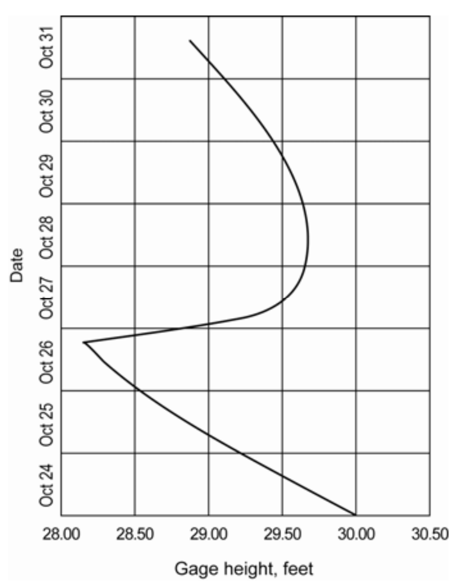
During the Engage phase, you examined graphs of streams' gage heights versus time. Sometimes, hydrologists are concerned about a particular gage height and when the gage for the stream measured that height. In those cases, it makes sense to consider a plot of time versus gage height.

Note to Leader: Recall that scatterplots of data are described as plots of the dependent variable versus the independent variable. In this example, gage height depends on time, so the scatterplot is described as gage height versus time.

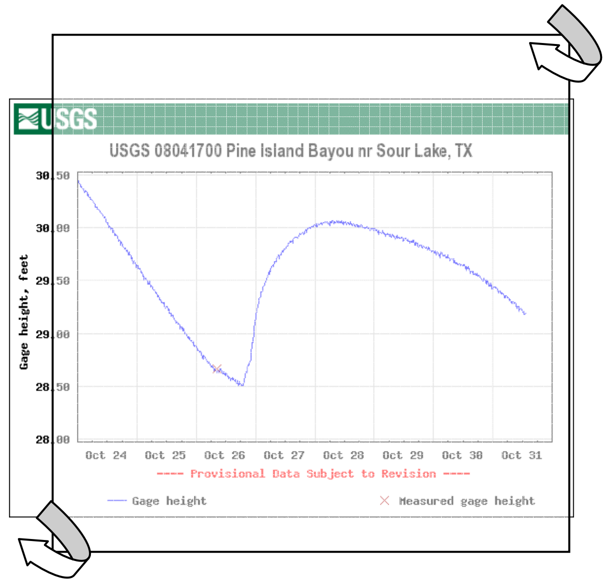
1. **For the scatterplot of gage height versus time, what are the inputs and outputs?**
   Time is the input and gage height is the output.

2. **Does this scatterplot represent a functional relationship? How can you tell?**
   Yes. For each x-value (time) there is only one corresponding y-value (gage height).

3. **What would the graph of the same data look like if we plotted time versus gage height? Sketch a prediction of the graph.**
   Responses may vary. Sample response:



Trace the graph of gage height versus time onto a sheet of patty paper. Be sure to trace the plot and the axes then label your axes on your patty paper.



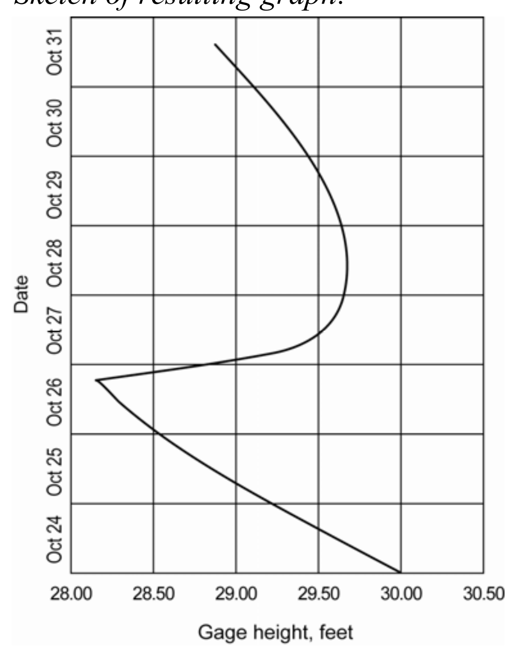
Take the top-right corner in your right hand and the bottom-left corner in your left hand and flip the patty paper over. Align the origin on the patty paper with the origin of your original graph and align the axes on the patty paper with the axes on the graph.

Note to Leader: One purpose of this part of the Explore phase is to describe in a non-numerical way the effect of generating an inverse relation from a given set of data. Participants may express concern that (1) the grid is not square; i.e., the gridlines from the *x*- and *y*-axes are not equidistant, and (2) there is no apparent origin of the graph. Assure them that for our purposes in this activity, it is OK.

Also, at this point some participants may question the absence of the vocabulary word "reflection" for this particular action. In this part of the Explore, we are developing the notion that the inverse is a reflection of a graph across the line $y = x$. The term "reflection" as it is used in this context will be formalized later.

4. **Sketch the resulting graph.**
   Responses may vary. Sample response:
   Patty paper reflection:

Sketch of resulting graph:



5. How is the new graph similar to the original graph? How are they different?
Responses may vary. Sample responses may include: The x- and y-axes changed places. The x-axis became the vertical axis. The y-axis became the horizontal axis. The x-axis became the gage height. The y-axis became the date. One graph represents a function, the other does not. The domain became the range, the range became the domain.

6. Is the new graph a function? How do you know? Given the situation, does it make sense for the graph to be or not to be a function? Explain your reasoning.
The new graph is not a function because there are multiple y-values for the same x-value (it fails the vertical line test). It makes sense that the new graph is not a function because there are various dates when the stream height is 29 feet, for example.

On the activity sheet, "Know When to Fold 'Em," trace each graph (both curves) onto a piece of patty paper. Be sure to also trace and label the axes.

Fold the patty paper to find all possible lines of symmetry for each pair of graphs shown. Identify the equation for each possible line of symmetry. Record the number of lines of symmetry and their equations for each graph on your recording sheet.

Possible responses:

	Total Number of Lines of Symmetry	Equation(s) of line(s) of symmetry
Graph 1	*2*	$y = x$ *and* $y = -x - 5$
Graph 2	*1*	$y = x$
Graph 3	*1*	$y = x$
Graph 4	*1*	$y = x$
Graph 5	*1*	$y = x$
Graph 6	*1*	$y = x$

7. What patterns do you notice among the lines of symmetry for each of the graphs?
Responses may vary. Participants should notice that for each of the given pairs of graphs, the line $y = x$ *was a line of symmetry.*

8. Which transformation describes the folds across a line of symmetry for your graphs?
Each fold across a line of symmetry reveals a reflection across the line of symmetry.

9. Make a summary statement describing the relationship between each pair of curves in the set of given graphs.
Responses may vary. Participants should observe that the pair of curves on each graph is a reflection across the line $y = x$*.*

Facilitation Question

- How are the curves related in terms of their lines of symmetry?
  Each curve is a reflection of the other across the line of symmetry ($y = x$*).*

Note to Leader: Discuss participants' summary statements as a whole group before proceeding to Part 2.

Facilitation Questions

- How are the curves related graphically?
  Each curve is a reflection of the other across the line of symmetry ($y = x$).
- What happens to the x-axis and y-axis when the graphs are reflected across their line(s) of symmetry?
  The x-axis becomes a vertical axis and the y-axis becomes a horizontal axis.

<u>Part 2</u>: Numerical and Symbolic Representations

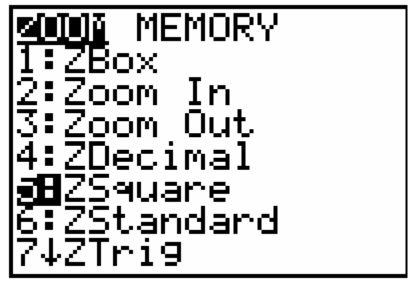
Note to Leader: At the end of Part 2, participants should understand that inverse relations have domains and ranges that are reversed (i.e., the domain of one relation is the range of the other and the range of one relation is the domain of the other). Participants should connect tabular (numerical), mapping, graphical, and symbolic representations of inverse relations.

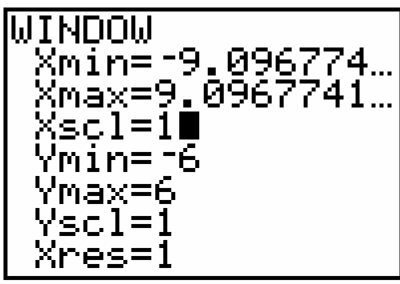
The following coordinates were used to create a geometric design for a quilting pattern.

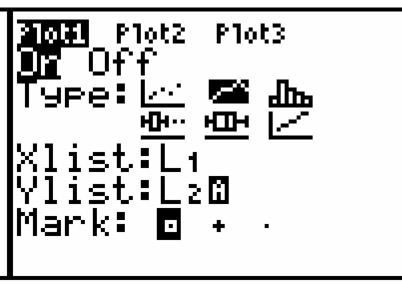
List 1	1	1	0	0	−1	−1	−2	−1	−2	−1	1
List 2	2	4	5	4	5	4	4	3	3	2	2

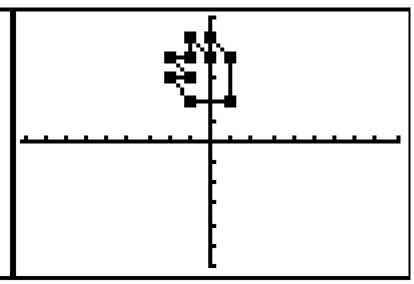
1. **Create a connected scatterplot of L_2 versus L_1. Sketch your graph and describe your viewing window.**

   TIP: Be sure to use a square window. After selecting your appropriate domain and range, use the Zoom-Square feature to square out the grid in your viewing window.







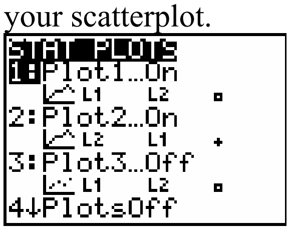


Facilitation Question
• What are the independent and dependent variables in this situation? *Though there is no clear dependency relationship, we are treating List 1 as independent and List 2 as dependent.*

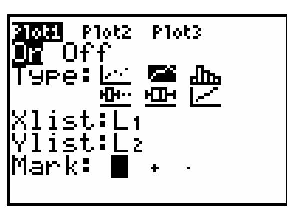
Technology Facilitation Tip:

A connected scatterplot can be created from the STAT PLOT menu:

Press [2nd][Y=] to select STAT PLOT. Use the [▲][▼] arrow keys to select your scatterplot.

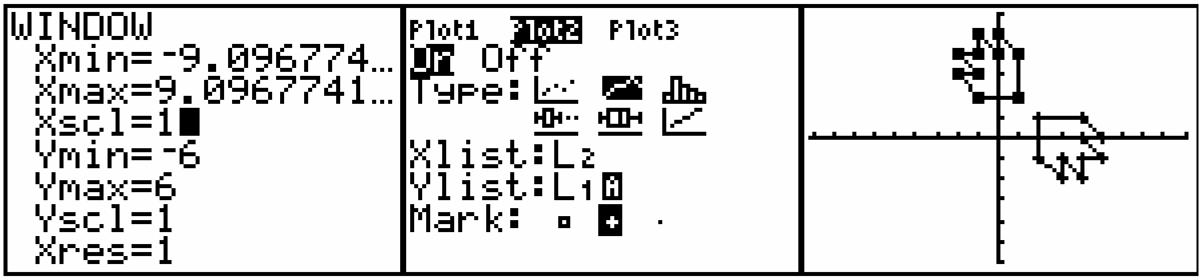


Use the [▲][▼] arrow keys to change the Type to connected-dot.

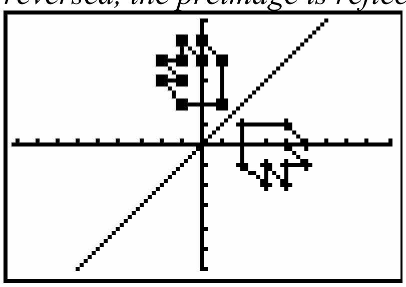


2. **What do you think would happen to the graph if we reversed the *x*- and *y*-values?**
   The graph would reflect across the line y = x.

3. Create a second connected scatterplot of L_1 versus L_2. Use a different plot symbol for this scatterplot. Graph this scatterplot with your original scatterplot. Sketch your graph and describe your viewing window. You may need to re-square your viewing window.



4. Compare the two scatterplots. How are they alike? How are they different?
Responses may vary. Participants should observe that when the x- and y-coordinates are reversed, the preimage is reflected across the line y = x.



Facilitation Questions

- Are the two images congruent or similar? How can you tell?
  The images appear to be both congruent and similar (recall that similarity is a special case of congruence with a scale factor of 1).

- What type of transformation might generate the second image from the first?
  A reflection across the line y= x.

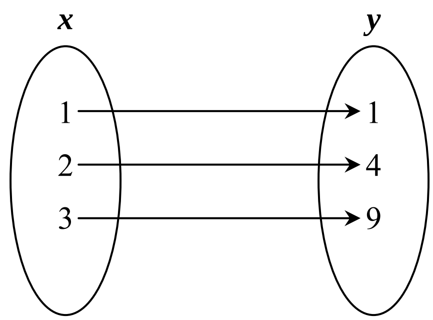
5. How are the *x*- and *y*-coordinates related from the first scatterplot to the second scatterplot? How could you represent this relationship symbolically?
The x-coordinates from the first scatterplot become the y-coordinates for the second scatterplot and the y-coordinates from the first scatterplot become the x-coordinates for the second scatterplot.

Symbolically, this relationship can be represented using the transformation mapping $T:(x, y) \to (y, x)$

Facilitation Question

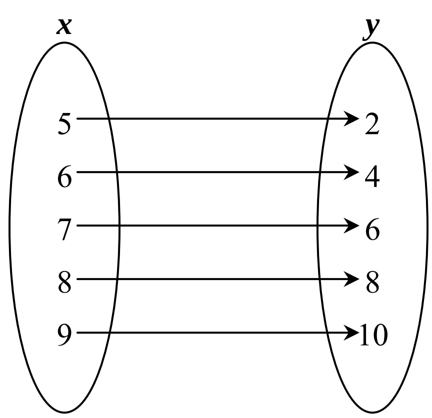
- How can we represent the reversal of *x* and *y* with a transformation mapping?
  $x \to y$ and $y \to x$

Recall that a mapping shows how domain elements for a relation relate, or "map to," their corresponding range elements. For example, the following mapping shows how the *x*-values {1, 2, 3} map to their corresponding *y*-values {1, 4, 9} for the function $y = x^2$.

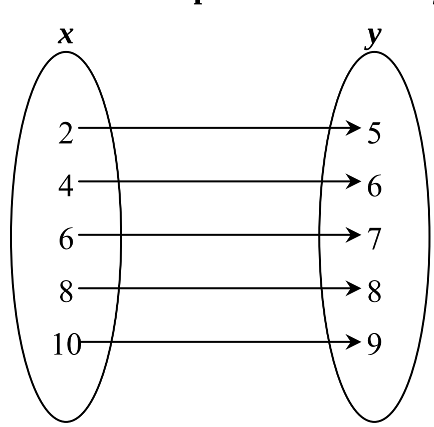


Enter the functions Y1 = 2*x* – 8 and $\mathbf{Y2 = \frac{1}{2}x + 4}$ into your graphing calculator.

6. **Use the table feature of your graphing calculator to generate values for a mapping for Y1 to show the replacement set for *y* when *x* = {5, 6, 7, 8, 9}.**
   Sample Response:



7. **Generate a mapping for Y2 to show the replacement set for *y* when *x* = {2, 4, 6, 8, 10}.**



8. How are the two mappings related?
The domain and the range for the two mappings are reversed.

9. In each mapping, to how many *y*-values does any given *x*-value map? Would you expect this to be true for other domain and range elements? How do you know?
Each x-value maps to only one y-value. This would be true for other domain and range elements as well.

10. What does this reveal about the relationships in each mapping?
That each x-value maps to only one y-value confirms that the relationships are functional.

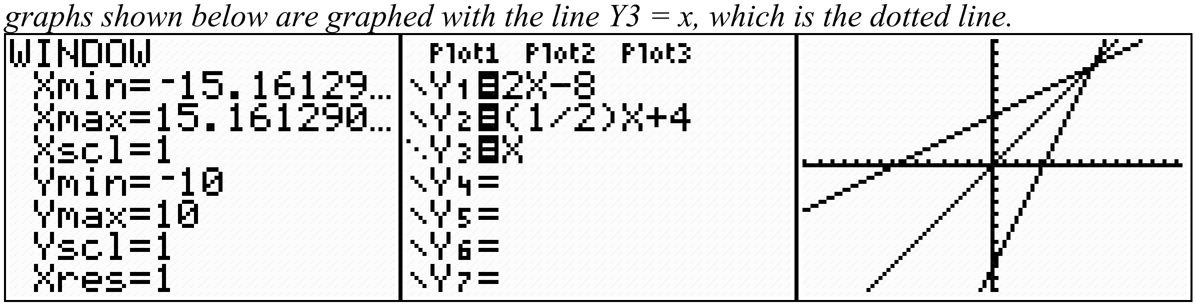
Facilitation Question
- What does it mean about a mathematical relationship when every *x*-value corresponds with only one *y*-value?
  Such a correspondence tells us that the relationship is a function.

11. In each mapping, how many *x*-values map to any given *y*-value? Would you expect this to be true for other domain and range elements? How do you know?
Only one x-value maps to any given y-value. This would be true for other domain and range elements as well.

12. What does this reveal about the relationships in each mapping?
When only one x-value maps to any given y-value, the relationship is said to be one-to-one.

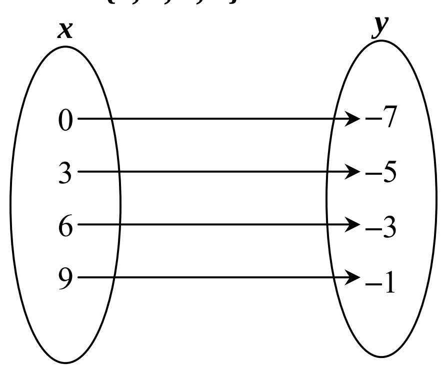
Facilitation Question
- What does it mean about a mathematical relationship when every *y*-value corresponds to only one *x*-value?
  Such a correspondence indicates a one-to-one correspondence, meaning that if you reversed the domain and range, the resulting relation would still be a function.

13. Examine the graphs of Y1 and Y2. How are they related? (Hint: Be sure you are using a square viewing window.)
The graphs of Y1 and Y2 appear to be reflections of one another across the line y = x. The graphs shown below are graphed with the line Y3 = x, which is the dotted line.

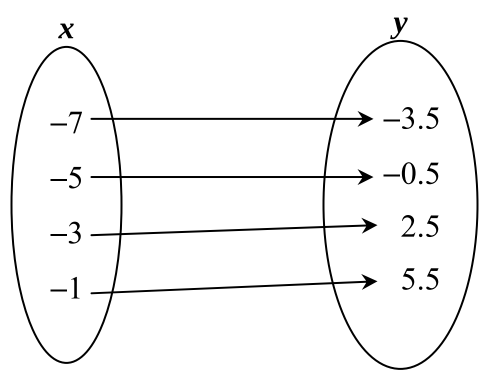


Enter the functions $Y_3 = \frac{2}{3}x - 7$ and $Y_4 = \frac{3}{2}x + 7$ into your graphing calculator.

14. Use the table feature of your graphing calculator to generate a mapping for Y3 to show the replacement set for *y* when *x* = {0, 3, 6, 9}.



15. Use the table feature of your graphing calculator to generate a mapping for Y4 to show the replacement set for *y* when *x* = {–7, –5, –3, –1}.

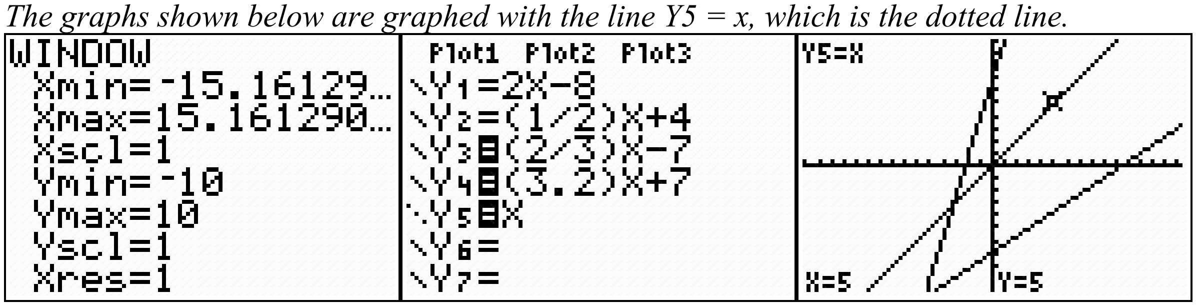


16. How are the two mappings related?

The domain and range are not reversed in these two mappings.

17. Examine the graphs of Y3 and Y4. How are they related? (Hint: Be sure you are using a square viewing window.)

The graphs of Y3 and Y4 do not appear to be reflections of one another across the line y = x. The graphs shown below are graphed with the line Y5 = x, which is the dotted line.



18. The functions in Y1 and Y2 are called "inverse relations" whereas the functions in Y3 and Y4 are not. Based on your mappings, graphs, and equations, why might this be the case?

Inverse relations have domains and ranges that are reversed, as revealed in the mappings. The graphs of inverse relations are reflections across the line $y = x$.

19. Based on your response to the previous question, how might we describe inverse relations graphically, numerically, and symbolically?

Graphically, inverse relations are reflections of each other across the line $y = x$.
Numerically, the domain and range of one relation become the range and domain of the second relation.
Symbolically, x (which represents the domain values) and y (which represents the range values) are interchanged.

Facilitation Questions

- How do the graphical and numerical representations of inverse relations connect to each other?
  Numerically, the x-values and y-values in inverse relations are reversed. Graphically, this reversal results in a reflection of each relation across the line $y = x$.

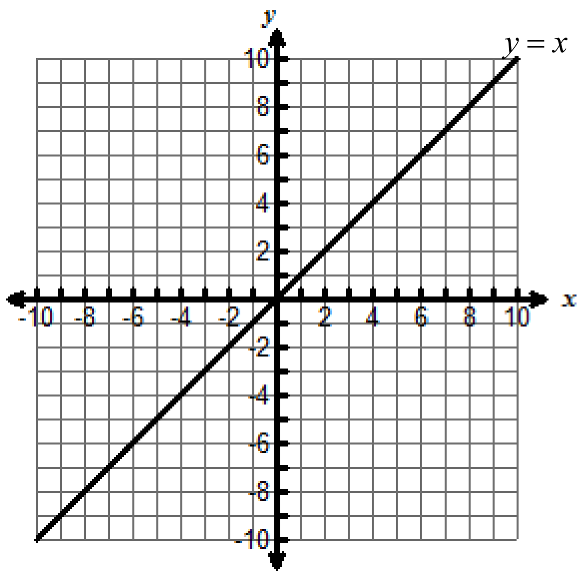
- How do the numerical and symbolic representations of inverse relations connect to each other?
  The symbols x and y are used to represent all domain elements and range elements, respectively. Using a symbol to represent all domain or range elements generalizes the reversal of all domain and range elements to generate an inverse relation.

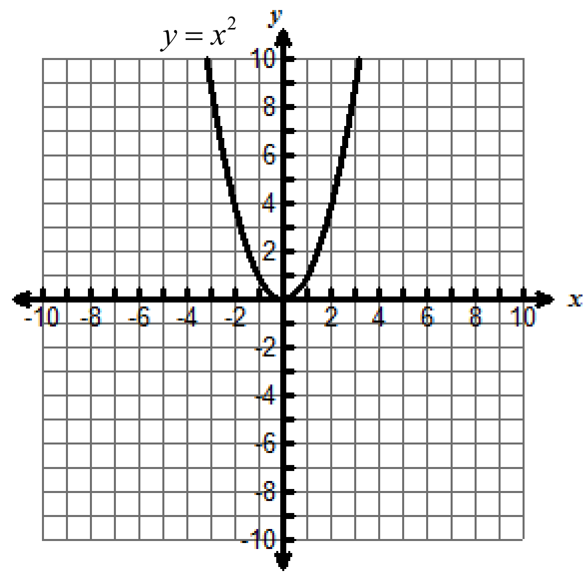
Note to Leader: Discuss participants' responses to the last question before proceeding to Part 3.

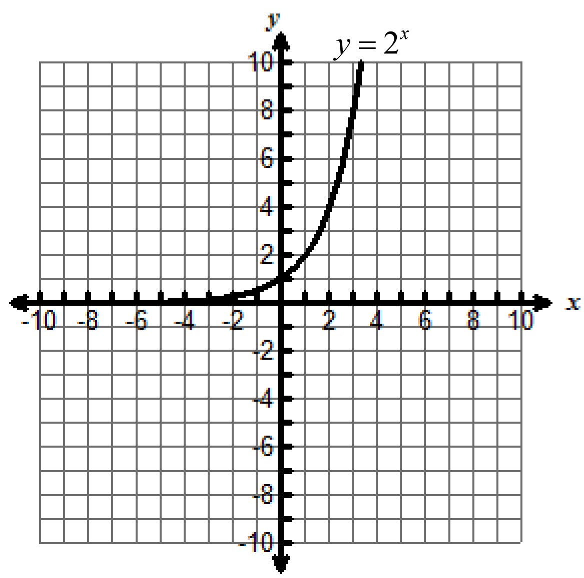
<u>Part 3</u>: Investigating Linear Functions
Notes to Leader: At the end of Part 3, participants should be able to determine the inverse of a linear function and describe an inverse relation as both a set of inverse number operations and a set of inverse transformations.

It is suggested that all participants do Part 3 (linear functions) together. Then, participants can jigsaw Part 4 (quadratic functions) and Part 5 (exponential functions) and discuss the similarities and differences among all three families of functions.

In Algebra 1, students investigate linear, quadratic, and exponential functions. The graphs of the parent functions are shown.

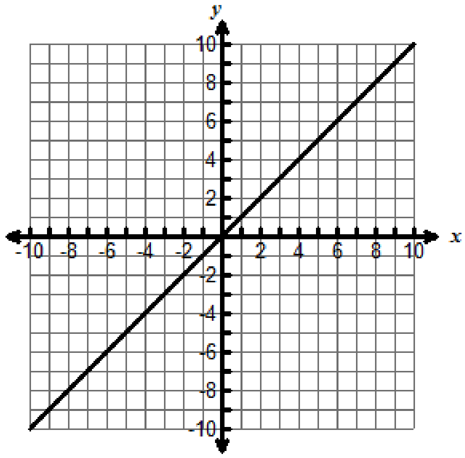






Trace each parent function onto a separate piece of patty paper. Be sure to trace and label the axes as well.

1. **Reflect the linear parent function across the line $y = x$. Sketch your resulting graph.**



2. **What is the domain and range of the inverse of the linear parent function? How do they compare with the original function?**
   The domain and range are both all real numbers, just like the original parent function.

3. **Is the inverse of the linear parent function also a function? How do you know?**
   Yes, the inverse is also a function since for each x-value there is only one corresponding y-value.

4. **What kind of function is the inverse of a linear function?**
   The inverse is also a linear function.

5. **What is the inverse of the function $y = \frac{2}{5}x - 7$? Find the inverse using at least two different methods.**
   $y = \frac{5}{2}(x+7)$ *or* $y = \frac{5}{2}x + \frac{35}{2}$

6. **How did you determine the inverse?**
   Participants could reflect the graph of the function across the line y = x then determine the equation of the reflected line.

   Symbolically, participants may reverse the domain (x) and range (y) then solve for y using inverse operations.

7. **What concepts and procedures did you apply to determine the inverse?**
   The concept of "inverse of a function," where the domain and range are reversed is applied.

   The procedures applied will vary depending on how participants found the inverse. Graphically, reflecting across the line y = x and determining the equation of a line from a *graph could be applied. Symbolically, applying inverse operations (in this case, addition then division) to solve for y could be applied.*

8. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{2}{5}x - 7$?

Multiply by $\frac{2}{5}$

Subtract 7

9. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{2}{5}x - 7$?

Add 7

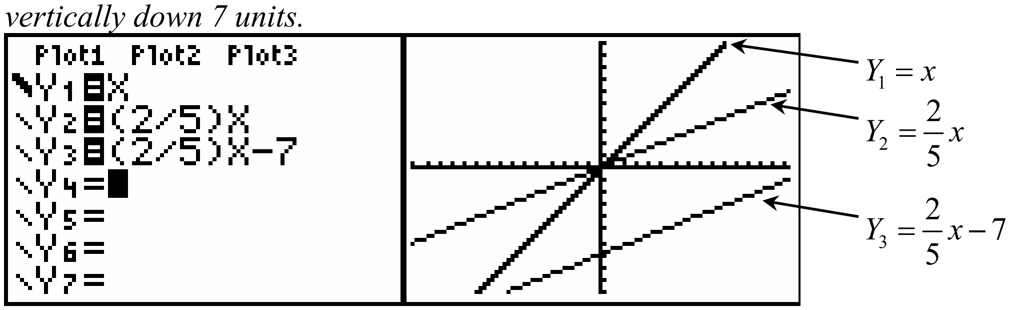
Divide by $\frac{2}{5}$

Facilitation Question • How are multiplication and division of fractions related? *Division by a fraction is symbolically equivalent to multiplying by the reciprocal.*

10. How do these two sets of operations compare?
The operations done to the inverse function are the inverse operations from what is done to the original function. They are also done in reverse order.

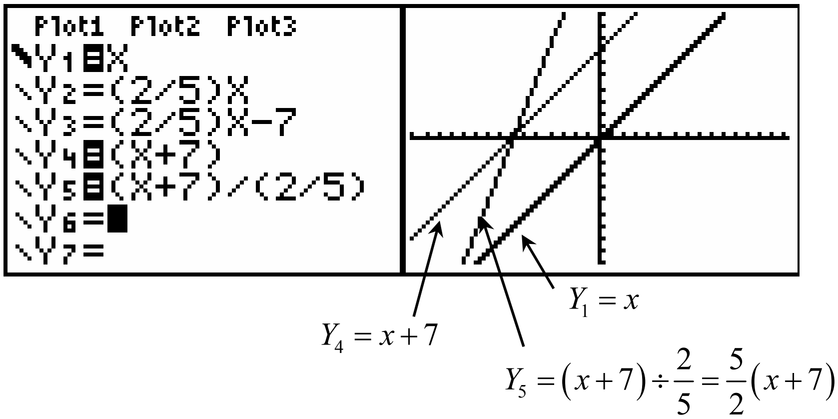
11. Describe the graph of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent function.

The parent function, y = x, is compressed vertically by a factor of $\frac{2}{5}$ *then translated vertically down 7 units.*



12. Describe the graph of the inverse of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent function.

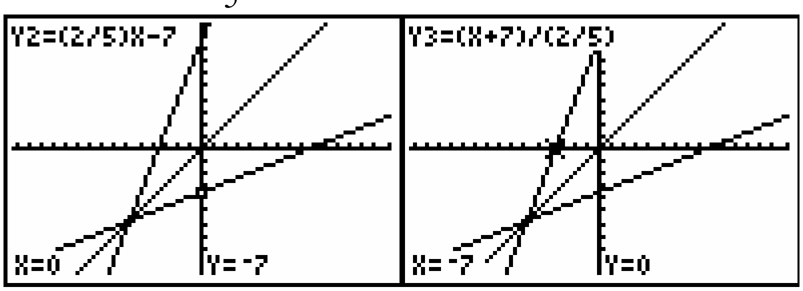
The parent function is shifted 7 units to the left then stretched vertically by a scale factor of $\frac{5}{2}$.



13. Compare the graphs of $y = \frac{2}{5}x - 7$, its inverse, and the line $y = x$.

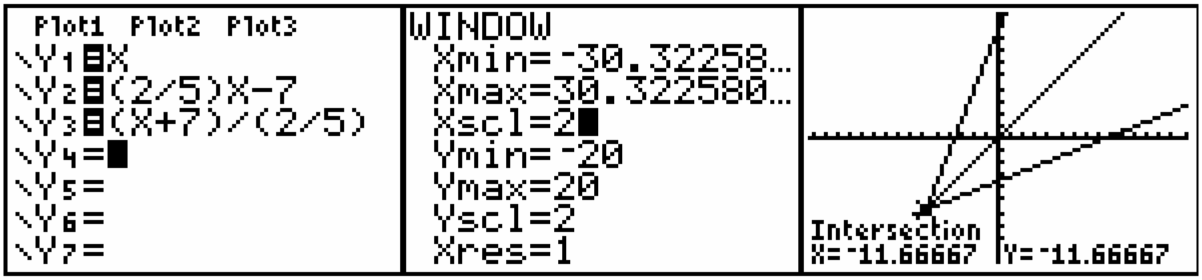
a. What do you notice about the intercepts of the graphs?

The y-intercept of $y = \frac{2}{5}x - 7$ *is the same as the x-intercept of its inverse. Likewise, the x-intercept of* $y = \frac{2}{5}x - 7$ *is the same as the y-intercept of its inverse.*



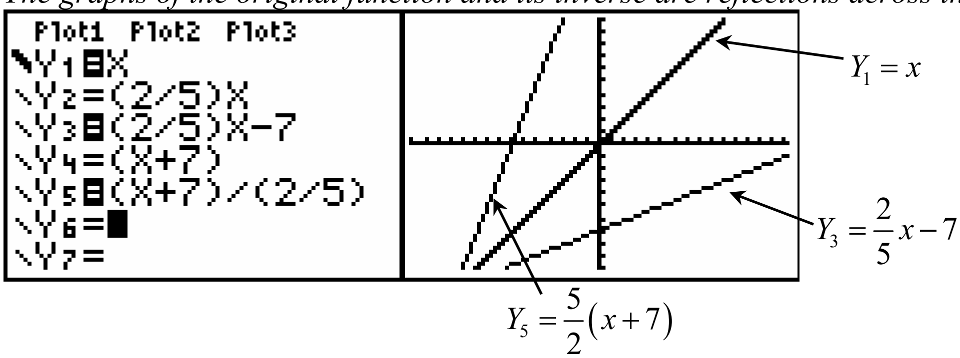
b. Where are the three graphs concurrent? What is the significance of this point?

The three graphs are concurrent at $\left(-11\frac{2}{3}, -11\frac{2}{3}\right)$. *This is the point where the graph of the original function intersects its inverse. This point is also located on the line of reflection.*



c. In terms of transformations, how do the graphs of the original function and its inverse compare?

The graphs of the original function and its inverse are reflections across the line y = x.



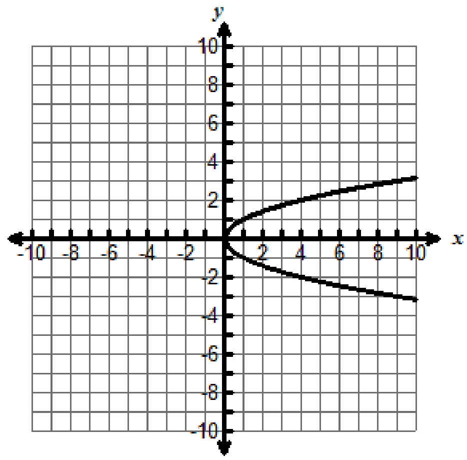
Also, transformations to the inverse relation are the inverse of the transformations (reverse transformations in reverse order) done to the original relation.

Note to Leader: Before moving on to Parts 4 and 5, be sure to discuss participants' responses to the last question. Draw out the common understandings of inverses as a set of inverse operations and inverse transformations.

<u>Part 4</u>: Quadratic Functions
Notes to Leader: At the end of Part 4, participants should be able to determine the inverse of a quadratic function (including restricting the range to result in a function) and connect inverse relations from a number operations perspective, a transformational perspective, and a symbolic perspective.

After doing Part 3 (linear functions) together, it is suggested that participants jigsaw Part 4 (quadratic functions) and Part 5 (exponential functions) then discuss the similarities and differences among all three families of functions.

1. **Reflect the quadratic parent function across the line *y* = *x*. Sketch your resulting graph.**



2. **What is the domain and range of the inverse of the quadratic parent function? How do they compare with the original function?**
   The domain is {x: x ≥ 0}, the range is all real numbers.
   The domain of the original parent function became the range of the inverse.
   The range of the original parent function became the domain of the inverse.

3. **Is the inverse of the quadratic parent function also a function? How do you know?**
   No. For most x-values (all except x = 0), there are two corresponding y-values. For example, when x = 4, y = +2 and −2.

4. **If the inverse is not a function, how can we restrict the domain and/or range of the original function so that the inverse is also a function?**
   If we only consider the domain {x: x ≥ 0}, then the inverse becomes a function.

5. **What kind of function is the inverse (range restricted) of a quadratic function?**
   The inverse of a quadratic function is a square root function.

6. Is either the original parent function or its inverse (range restricted) one-to-one? How do you know?
The parent function $y = x^2$ is not one-to-one since for all y-values except y = 0, there are two x-values yielding the same y-value. The inverse function (range restricted) is one-to-one since every y-value is only generated by one x-value.

7. What is the inverse function of the function $y = -3(x-1)^2 + 2$? Find the inverse using at least two different methods.
$y = 1 + \sqrt{\frac{x-2}{-3}}$ *or* $y = 1 - \sqrt{\frac{x-2}{-3}}$
Typically, the principal (positive) square root is used for square root functions.

8. How did you determine the inverse?
Graphically, participants could reflect the graph of the function across the line y = x then determine the equation of the reflected curve.

Symbolically, participants may reverse the domain (x) and range (y) then solve for y using inverse operations.

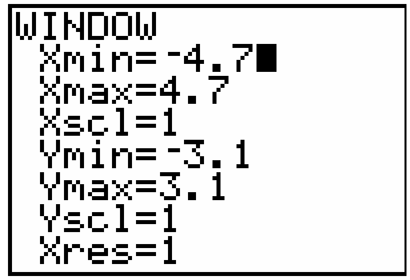
9. What concepts and procedures did you apply to determine the inverse?
The concept of "inverse of a function," where the domain and range are reversed, is applied.

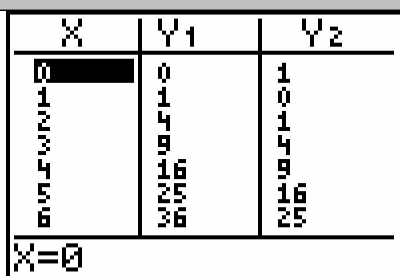
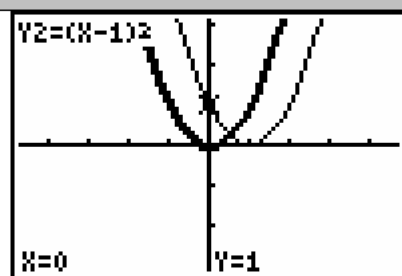
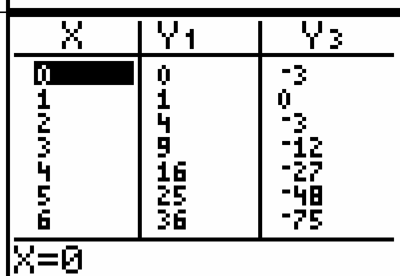
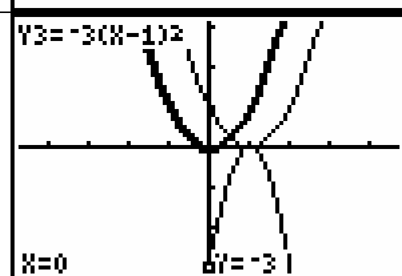
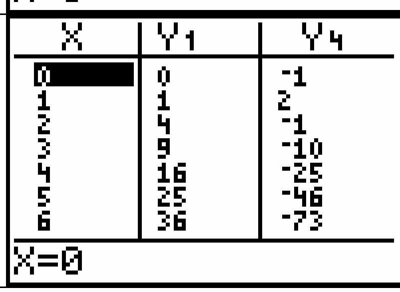
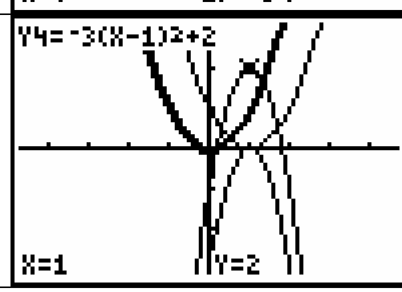
The procedures applied will vary depending on how participants found the inverse. Graphically, reflecting across the line y = x and determining the equation of the curve from a graph could be applied. Symbolically, applying inverse operations (in this case, subtraction, division, square rooting, then addition) to solve for y could be applied.

10. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = -3(x-1)^2 + 2$?
Subtract 1
Square the value
Multiply by −3
Add 2

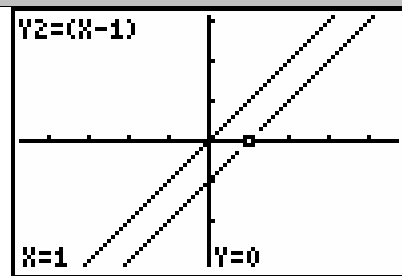
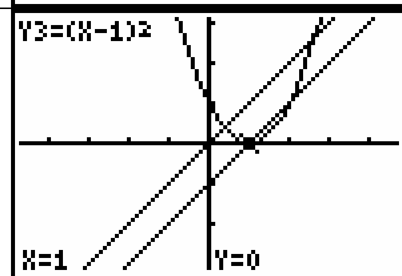
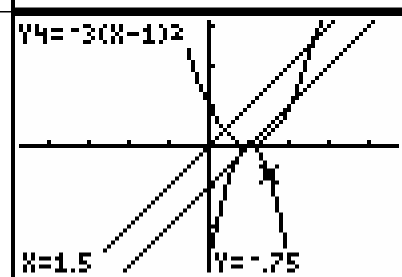
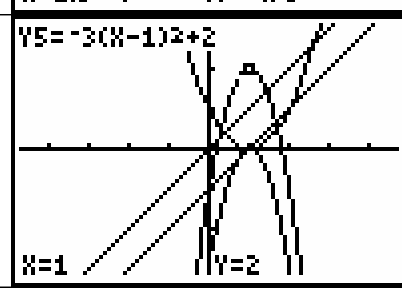
11. From the perspectives of number operations and transformations, develop the quadratic function, $y=-3(x-1)^2+2$, from the parent function $y = x^2$. Include tabular, graphical, and symbolic representations of the number operation as it applies to the function.

Note to Leader: *All graphs are shown using the viewing window at right.*



Number Operation	Tabular	Graphical	Symbolic
Subtract 1 from the *x*-value in the parent function $y = x^2$.			$y=(x-1)^2$
Multiply by a factor of −3			$y=-3(x-1)^2$
Add 2			$y=-3(x-1)^2+2$

12. Now, use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the quadratic function, $y = -3(x-1)^2 + 2$, from $y = x$.

Number Operation	Tabular	Graphical	Symbolic
Subtract 1 from the parent function $y = x$	X Y1 Y2 0 0 -1 1 1 0 2 2 1 3 3 2 4 4 3 5 5 4 6 6 5 X=0		$y = x - 1$
Square the value (apply the parent function operation)	X Y2 Y3 0 -1 1 1 0 0 2 1 1 3 2 4 4 3 9 5 4 16 6 5 25 X=0		$y = (x-1)^2$
Multiply by a factor of -3	X Y3 Y4 0 1 -3 1 0 0 2 1 -3 3 4 -12 4 9 -27 5 16 -48 6 25 -75 X=0		$y = -3(x-1)^2$
Add 2	X Y4 Y5 0 -3 -1 1 0 2 2 -3 -1 3 -12 -10 4 -27 -25 5 -48 -46 6 -75 -73 X=0		$y = -3(x-1)^2 + 2$

13. How does each successive number operation transform the function numerically, graphically, and symbolically?

Responses may vary. Participants should connect the symbolic representation (general case) to the numeric representations contained in the table (specific cases).

Participants should also observe that the graphical representation is a set of points from the table that can be described generally by the symbolic representation. Each successive operation transforms the numeric/tabular values, and this transformation affects the symbolic (general rule) and graphical representations in turn.

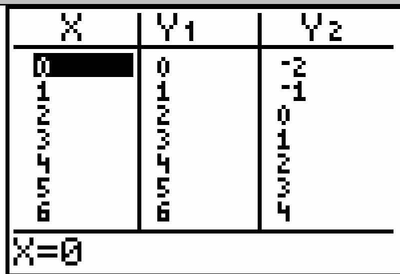
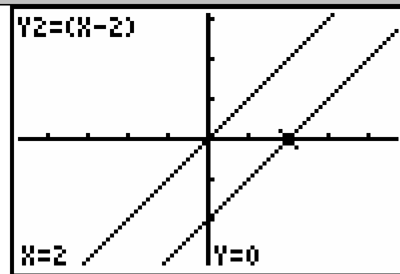
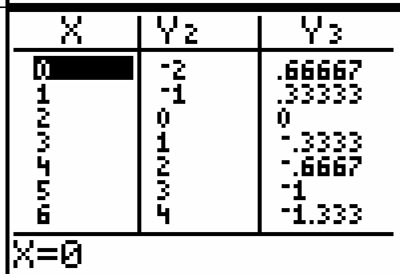
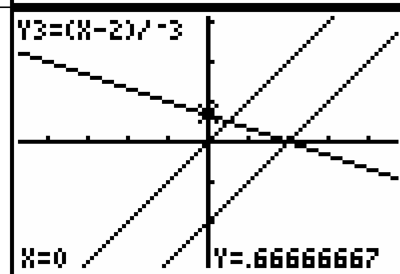
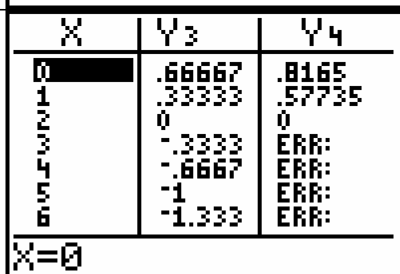
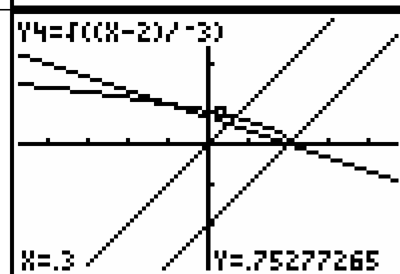
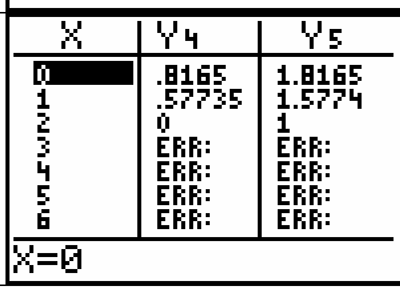
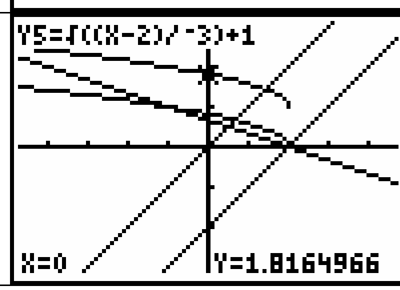
14. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = -3(x-1)^2 + 2$?

Subtract 2
Divide by −3
Take the square root
Add 1

15. How does this set of operations compare to the operations applied to generate the function $y = -3(x-1)^2 + 2$?

The operations done to the inverse function are the inverse operations from what is done to the original function. They are also done in reverse order.

16. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the square root inverse of the function, $y=-3(x-1)^2+2$, from $y = x$.

Number Operation	Tabular	Graphical	Symbolic
Subtract 2			$y=x-2$
Divide by −3			$y=\frac{x-2}{-3}$ *or* $y=-\frac{1}{3}(x-2)$
Take the square root			$y=\sqrt{\frac{x-2}{-3}}$ *or* $y=\sqrt{-\frac{1}{3}(x-2)}$
Add 1			$y=\sqrt{\frac{x-2}{-3}}+1$ *or* $y=1+\sqrt{-\frac{1}{3}(x-2)}$

17. How do the inverse operations relate to the inverse function?

The inverse function is a set of inverse operations performed in reverse order.

18. Generalize how inverse relations for quadratic functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.
Possible responses:

Number Operation	Tabular	Graphical	Symbolic
The number operations for inverse relations are the inverse operations that are applied to the original function, applied in reverse order.	*Range values from the table of the inverse relation match the domain values for the original function.*	*The graphs of a function and its inverse relation or function are reflections across the line y = x.*	*The inverse relation can be generated by reversing the domain and range elements (represented by x and y, respectively) then solving for y.*

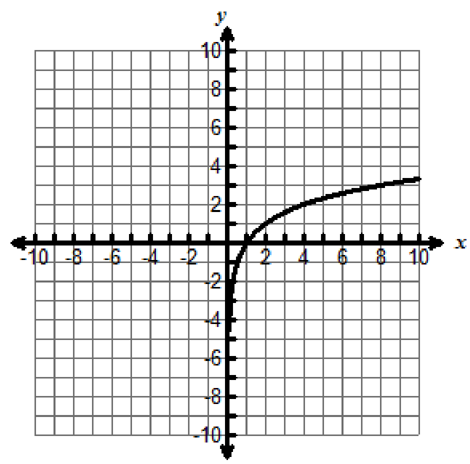
Note to Leader: When both groups come back together to debrief the jigsawed Parts 4 and 5, be sure to ask participants to discuss their responses to the last question of both parts.

Facilitation questions for the whole-group debrief of Parts 4 and 5 are at the end of Part 5.

<u>Part 5</u>: Exponential Functions
Notes to Leader: At the end of Part 5, participants should be able to determine the inverse of an exponential function (including restricting the range to result in a function) and connect inverse relations from a number operations perspective, a transformational perspective, and a symbolic perspective.

After doing Part 3 (linear functions) together, it is suggested that participants jigsaw Part 4 (quadratic functions) and Part 5 (exponential functions) then discuss the similarities and differences among all three families of functions.

1. **Reflect the exponential parent function, $y = 2^x$, across the line $y = x$. Sketch your resulting graph.**



2. **What is the domain and range of $y = 2^x$?**
The domain is all real numbers and the range is {y: y > 0}.

3. **What is the domain and range of the inverse of $y = 2^x$? How do they compare with the original function?**
The domain is {x: x > 0} and the range is all real numbers.
The domain of the original parent function became the range of the inverse.
The range of the original parent function became the domain of the inverse.

4. **What asymptote(s) does the original function, $y = 2^x$ have? Why does this asymptote exist?**
There is a horizontal asymptote at y = 0 (the x-axis) because there are no powers of 2 that yield a negative value.

5. **What asymptote(s) does the inverse of $y = 2^x$ have? How do they compare to the asymptotes of the original function?**
There is a vertical asymptote at x = 0 (the y-axis). This asymptote is a reflection of the original asymptote, y = 0, across the line y = x.

6. Is the inverse of $y = 2^x$ also a function? How do you know?
Yes, the inverse is also a function since for each x-value there is only one corresponding y-value.

7. What kind of function is the inverse of an exponential function?
The inverse of an exponential function is a logarithmic function.

8. Is either the original parent function or its inverse one-to-one? How do you know?
Both the parent function and the inverse function are one-to-one since every y-value is only generated by one x-value.

9. What is the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$? Find the inverse using at least two different methods.
$y = log\left(2\left(x-3\right)\right)+1$ *or* $y = log\left(2x-6\right)+1$

10. How did you determine the inverse?
Participants could reflect the graph of the function across the line y = x then determine the equation of the reflected curve.

Symbolically, participants may reverse the domain (x) and range (y) then solve for y using inverse operations.

11. What concepts and procedures did you apply to determine the inverse?
The concept of "inverse," where the domain and range are reversed is applied.

The procedures applied will vary depending on how participants found the inverse. Graphically, reflecting across the line y = x and determining the equation of the curve from a graph could be applied. Symbolically, applying inverse operations (in this case, subtraction, division, taking the logarithm, then addition) to solve for y could be applied.

12. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{1}{2}(10)^{x-1} + 3$?

Subtract 1
Raise 10 to that power
Multiply by $\frac{1}{2}$
Add 3

13. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$?

Subtract 3

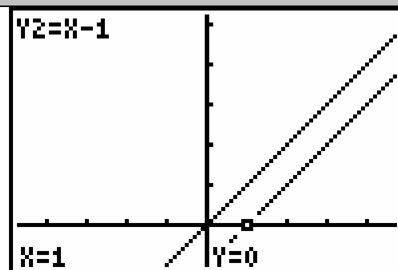
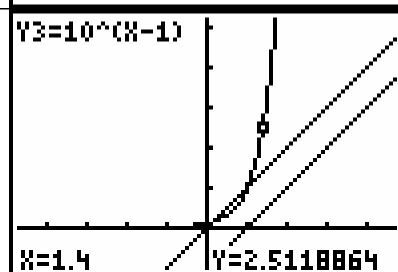
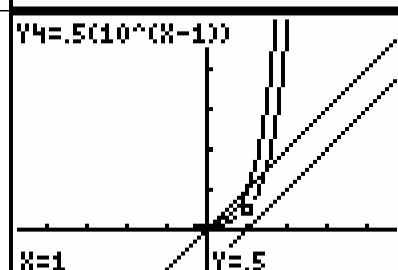
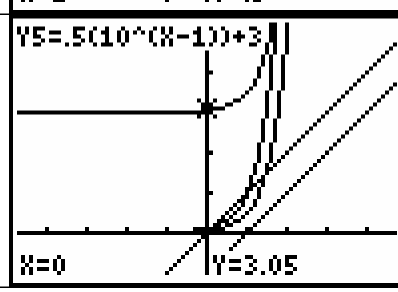
Multiply by 2 (divide by $\frac{1}{2}$)
Take the base-10 logarithm
Add 1

14. How do these two sets of operations compare?
The operations done to the inverse function are the inverse operations from what is done to the original function. They are also done in reverse order.

15. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the exponential function, $y = \frac{1}{2}(10)^{x-1} + 3$, from $y = x$.

Note to Leader: *All graphs are shown using the following viewing window:*

```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-1
 Ymax=5.2
 Yscl=1
 Xres=█
```

Number Operation	Tabular	Graphical	Symbolic
Subtract 1	X | Y1 | Y2 -3 | -3 | -4 -2 | -2 | -3 -1 | -1 | -2 0 | 0 | -1 1 | 1 | 0 2 | 2 | 1 3 | 3 | 2 X= -3		$y = x - 1$
Raise 10 to that power	X | Y2 | Y3 -3 | -4 | 1E -4 -2 | -3 | .001 -1 | -2 | .01 0 | -1 | .1 1 | 0 | 1 2 | 1 | 10 3 | 2 | 100 X=0		$y = 10^{x-1}$
Multiply by $\frac{1}{2}$	X | Y3 | Y4 -3 | 1E -4 | 5E -5 -2 | .001 | 5E -4 -1 | .01 | .005 0 | .1 | .05 1 | 1 | .5 2 | 10 | 5 3 | 100 | 50 X=0		$y = \frac{1}{2}\left(10^{x-1}\right)$
Add 3	X | Y4 | Y5 -3 | 5E -5 | 3.0001 -2 | 5E -4 | 3.0005 -1 | .005 | 3.005 0 | .05 | 3.05 1 | .5 | 3.5 2 | 5 | 8 3 | 50 | 53 X=0		$y = \frac{1}{2}\left(10^{x-1}\right) + 3$

16. How does each successive number operation transform the function numerically, graphically, and symbolically?

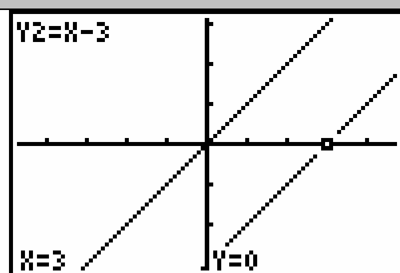
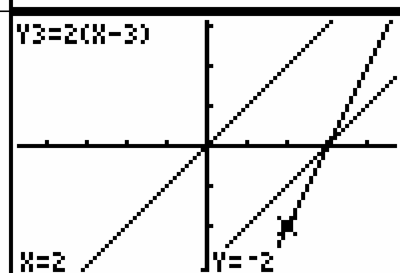
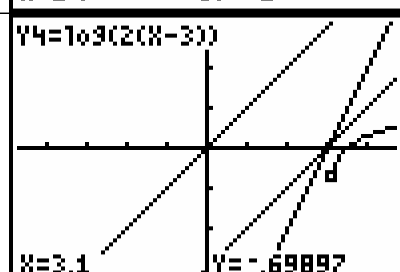
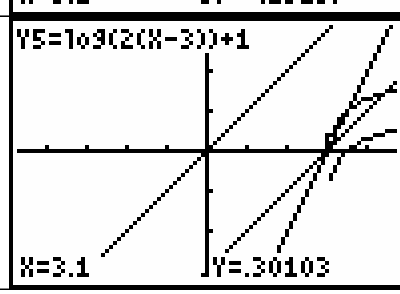
Responses may vary. Participants should connect the symbolic representation (general case) to the numeric representations contained in the table (specific cases).

Participants should also observe that the graphical representation is a set of points from the table that can be described generally by the symbolic representation. Each successive operation transforms the numeric/tabular values and this transformation affects the symbolic (general rule) and graphical representations in turn.

17. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the logarithm inverse of the function, $y=\frac{1}{2}(10)^{x-1}+3$, from $y = x$.

Note to Leader: *All graphs are shown using the following viewing window:*

```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-3.1
 Ymax=3.1
 Yscl=1
 Xres=■
```

Number Operation	Tabular	Graphical	Symbolic
Subtract 3	X | Y1 | Y2 0 | 0 | -3 1 | 1 | -2 2 | 2 | -1 3 | 3 | 0 4 | 4 | 1 5 | 5 | 2 6 | 6 | 3 X=0		$y = x - 3$
Multiply by 2 (divide by $\frac{1}{2}$)	X | Y2 | Y3 0 | -3 | -6 1 | -2 | -4 2 | -1 | -2 3 | 0 | 0 4 | 1 | 2 5 | 2 | 4 6 | 3 | 6 X=0		$y = 2(x-3)$
Take the base-10 logarithm	X | Y3 | Y4 0 | -6 | ERR: 1 | -4 | ERR: 2 | -2 | ERR: 3 | 0 | ERR: 4 | 2 | .30103 5 | 4 | .60206 6 | 6 | .77815 X=0		$y = log(2(x-3))$
Add 1	X | Y4 | Y5 0 | ERR: | ERR: 1 | ERR: | ERR: 2 | ERR: | ERR: 3 | ERR: | ERR: 4 | .30103 | 1.301 5 | .60206 | 1.6021 6 | .77815 | 1.7782 X=0		$y = log(2(x-3))+1$

18. How do the inverse operations relate to the inverse function?
The inverse function is a set of inverse operations performed in reverse order.

19. Generalize how inverse relations for exponential functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.
Possible responses:

Number Operation	Tabular	Graphical	Symbolic
The number operations for inverse relations are the inverse operations that are applied to the original function, applied in reverse order.	*Range values from the table of the inverse relation match the domain values for the original function.*	*The graphs of a function and its inverse relation or function are reflections across the line $y = x$.*	*The inverse relation can be generated by reversing the domain and range elements (represented by x and y, respectively) then solving for y.*

Use the Facilitation Questions to debrief the experiences in Parts 3, 4, and 5.

Facilitation Questions

- How are inverses of linear, quadratic, and exponential functions similar?
  The inverses are all reflections across the line y = x.
  Transformations of the parent function result in similar translations and vertical dilations.
  Attributes of x become attributes of y; e.g., x-intercepts of the original function become y-intercepts of the inverse relation and vice versa.

- How are inverses of linear, quadratic, and exponential functions different?
  Inverses of linear and exponential functions are functions while inverses of quadratic functions are not functions unless the range is restricted.

- How do different limitations on domain and range (maximum/minimum value, asymptotes, discontinuities, etc.) affect inverse relations/functions?
  A maximum/minimum value such as the vertex of a quadratic function results in a function that is not one-to-one. Inverses of these functions are non-functional relations. Asymptotes such as the horizontal asymptote in an exponential function are also reflected across the line y = x when the logarithmic inverse function is generated.

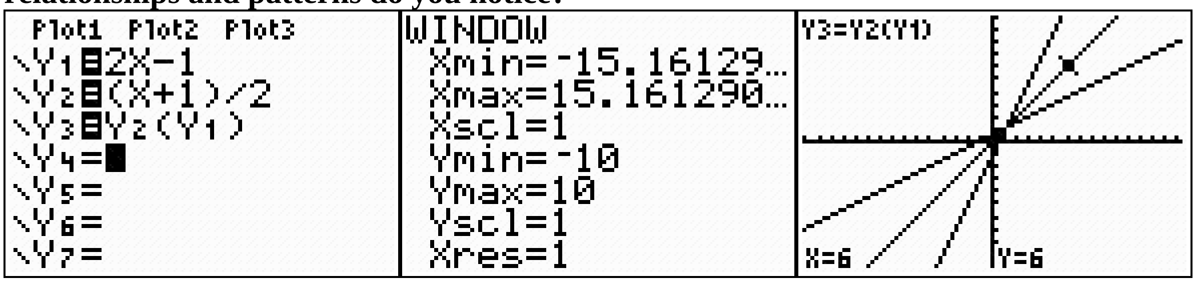
- What kinds of functions have inverses that are not functions without domain/range restrictions?
  Functions that are not one-to-one such as quadratic functions (or participants may mention absolute value or other functions depending on their prior experiences) have inverses that are not functions without domain/range restrictions.

- What kinds of functions have inverses that are functions without the need to restrict the domain/range?
  One-to-one functions such as linear or exponential have inverses without the need for domain/range restrictions.

<u>Part 6</u>: Compositions of Functions (Extension)
Note to Leader: At the end of Part 6, participants should recognize and describe attributes of composition of functions, including whether or not composition is commutative and how to account for any domain/range restrictions on the function or its inverse.

Enter the function *y* = 2*x* – 1 into *Y1* of your graphing calculator's function editor. Enter the inverse of this function into *Y2*. Enter the composition of *Y2* and *Y1* into *Y3*.

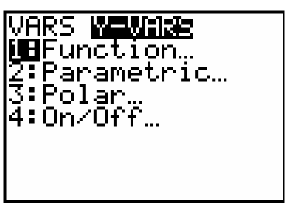
1. **Sketch the graphs of the three functions (describe your viewing window). What relationships and patterns do you notice?**



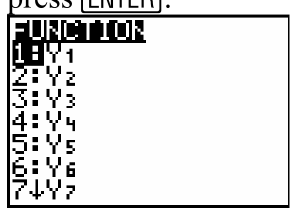
Participants may observe that the composition of the two functions is the line y = x.

Technology Facilitation Tip:
To find the y-variables as shown in the first screen shot above:

Press [VARS] to select the Variables menu. Use the [▸] arrow key to select *y*-variables (Y-VARS).

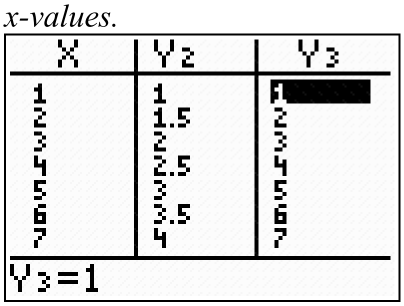


Press [1] to select FUNCTION variables. Use the [▴][▾] arrow keys to select the desired y-variable then press [ENTER].

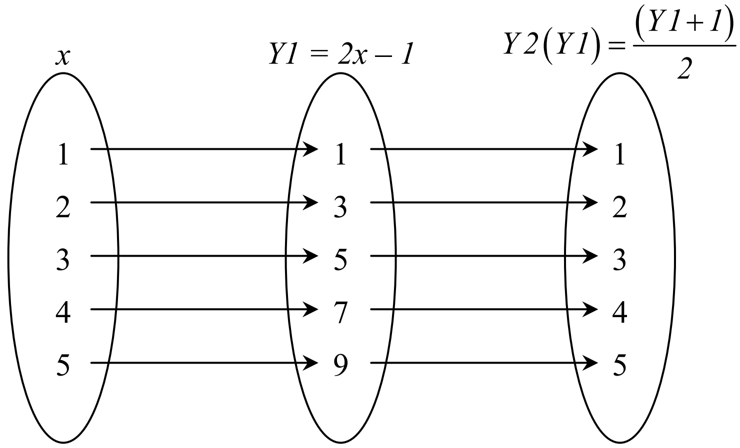


Function notation can be used in the function editor to select an input other than X. For example, to find Y2 values for a domain (input values) of Y1, type Y2(Y1) as shown in the first screen shot above.

2. **Look at the table values for each of the three functions. What do you notice?**
Responses may vary. Participants should notice that the values for Y3 are the same as the x-values.



3. How could you represent the composition of these two functions with a mapping?



4. What effect does composition of inverse functions have? Why do you think this is so?
Composition of inverse functions maps all domain values back onto themselves. This works because the first function performs arithmetic operations on the domain values then the inverse function "undoes" the arithmetic operations.

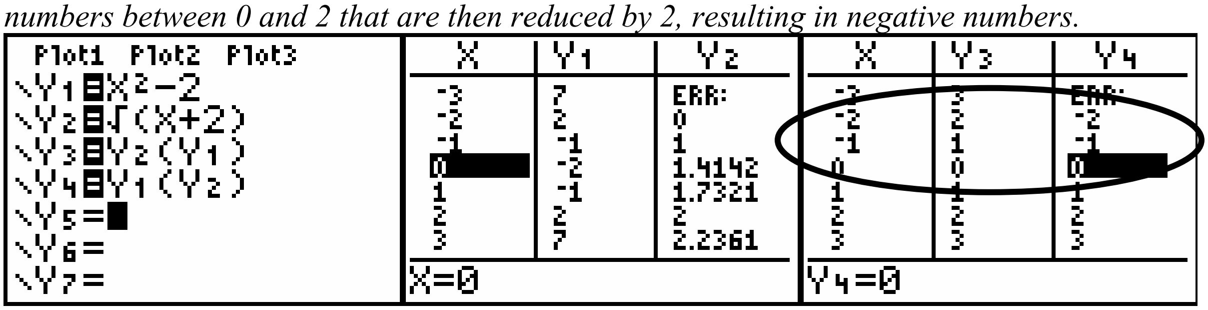
5. Does the order of composition matter? Explain your answer.
For linear functions and their inverses, the order does not matter because there are no domain or range restrictions.

Note to Leader: However, if two inverses have domain or range restrictions (such as the case with quadratic functions and their inverse square-root functions which will be investigated momentarily), the compositions could yield different subsets of real numbers, depending on which function operated on the set of real numbers first.

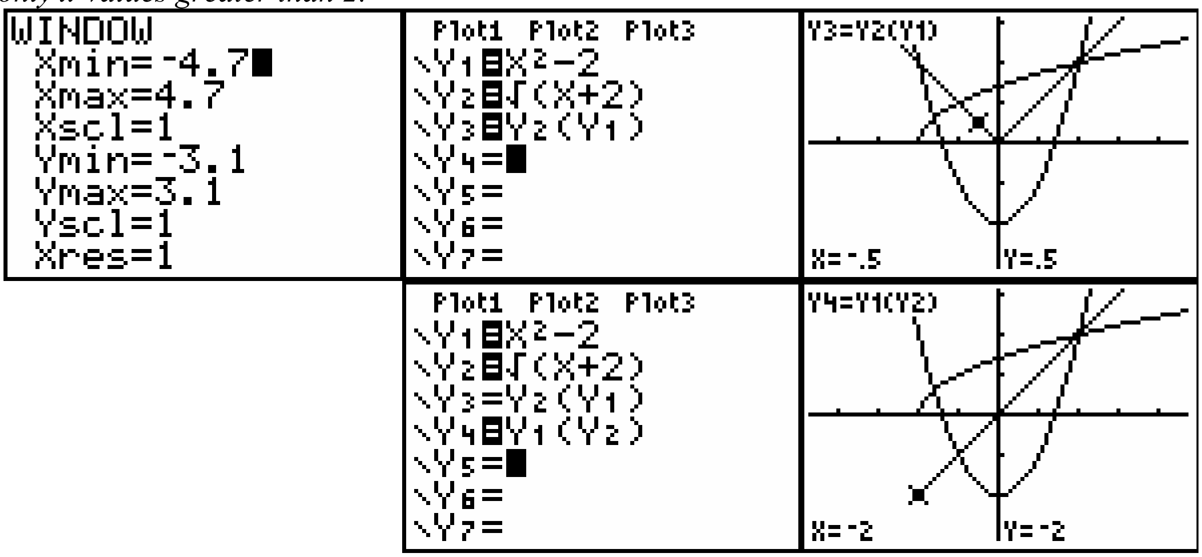
After participants complete the next prompt, be sure to ask them to discuss their findings.

6. Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse <u>function</u>. Describe the result numerically, graphically, and symbolically.

Numerically, neither composition returns the original x-values. Further, the order of composition yields a different set of y-values for $\{x : -2 \le x < 0\}$. Over this domain, the composition $Y2 \circ Y1$ yields the square root of a collection of positive numbers. Thus, the result is always positive. Meanwhile, the composition $Y1 \circ Y2$ yields a collection of positive numbers between 0 and 2 that are then reduced by 2, resulting in negative numbers.



Graphically, the composition of $Y2 \circ Y1$ yields the graph of the absolute value parent function, $y = |x|$. The composition of $Y1 \circ Y2$ yields the linear parent function, y = x, restricted to the domain x ≥ 2. Interestingly, the domain restriction corresponds to the range restriction on $Y2 = \sqrt{x+2}$ because the range of Y2 becomes the domain of Y1, consisting of only x-values greater than 2.



Symbolically, $Y1 = f(x) = x^2 - 2$, $Y2 = g(x) = \sqrt{x+2}$, $Y3 = Y2 \circ Y1 = g \circ f(x)$, and $Y4 = Y1 \circ Y2 = f \circ g(x)$. The composition functions reduce to:

$$\begin{aligned} g \circ f(x) &= \sqrt{(x^2-2)+2} \\ &= \sqrt{x^2} \\ &= |x| \end{aligned} \qquad \begin{aligned} f \circ g(x) &= \left(\sqrt{x+2}\right)^2 - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Note to Leader: Discuss participants' responses before proceeding to Part 7.

Facilitation Questions

- What do the different representations of the composition of a quadratic function and its inverse function reveal? How are those revelations related?
  The tabular/numerical representations show specific cases of input-output for both compositions. The graphical representations spatially show how the two compositions are related to each other and to the original and inverse functions. The symbolic representations show the subtle yet important differences between the two compositions $f \circ g$ and $g \circ f$.

- Based on your experiences with quadratic functions, what would you expect compositions of other types of functions and their inverses to be like? Why?
  Responses may vary. Participants may conjecture that exponential functions would behave more like linear functions in that they are both continuously increasing or decreasing functions.

<u>Part 7</u>: Extension of Compositions

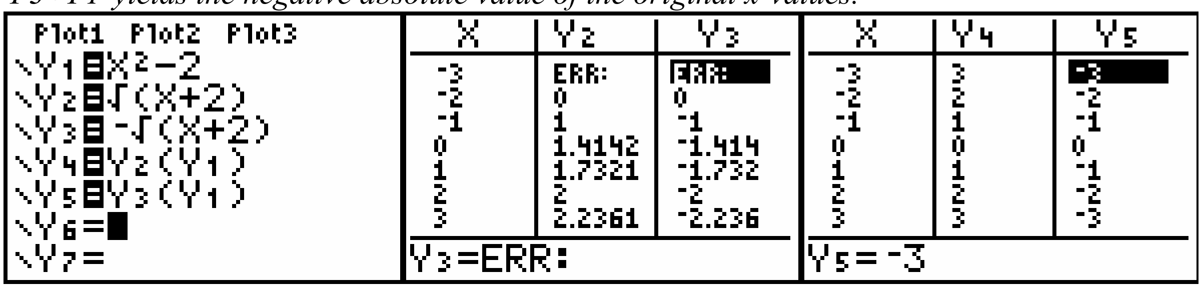
Note to Leader: Part 7 is an optional extension of compositions from Part 6. It is designed to allow participants to extend their understanding of compositions of inverse relations and functions. If participants do not experience Part 7, you can still have a meaningful discussion during the Explain and Elaborate phases.

1. **Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse <u>relation</u>. Describe the result numerically, graphically, and symbolically.**

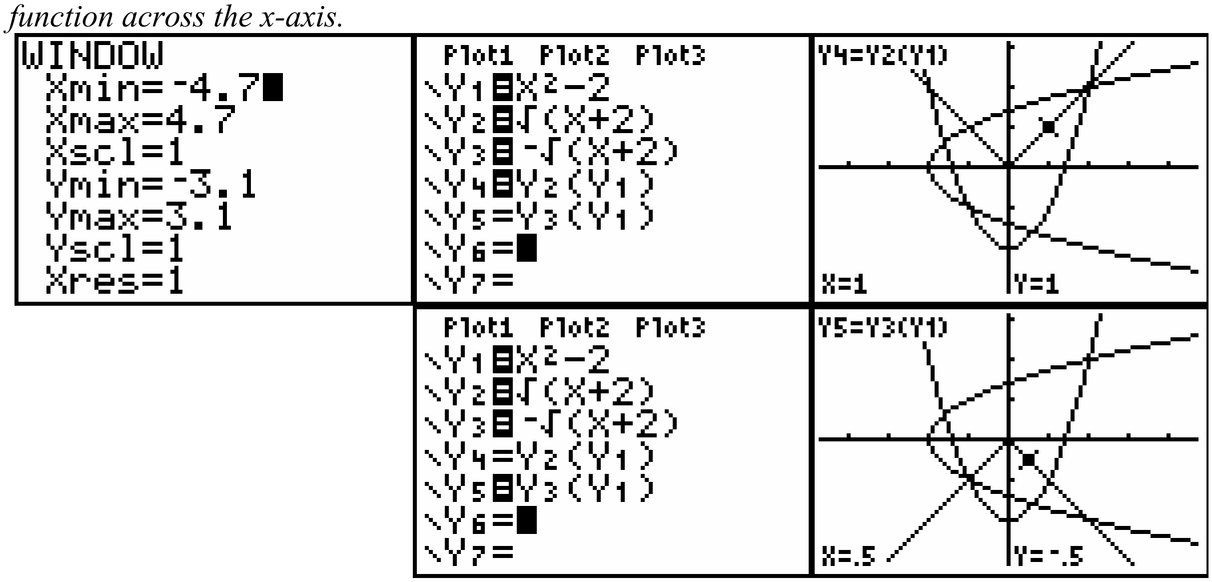
   Note to Leader: *Because the inverse <u>relation</u> of $y = x^2 - 2$ is a parabola opening up along the positive x-axis, we must plot the positive square root and the negative square root as two separate functions. To make screen shots and graphical analysis easier, we will consider $g \circ f$ first then explore $f \circ g$.*

   Case 1: Inverse is not a function – Composition of Inverse(Original)

   Numerically, the composition $Y2 \circ Y1$ yields the absolute value of the original x-values and $Y3 \circ Y1$ yields the negative absolute value of the original x-values.



   Graphically, the composition of $Y2 \circ Y1$ yields the graph of the absolute value parent function, $y = |x|$. The composition of $Y3 \circ Y1$ yields a reflection of the absolute value parent function across the x-axis.

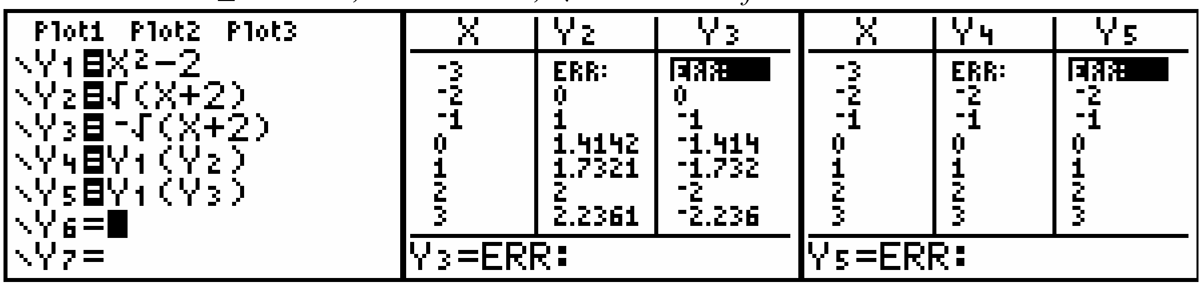


Symbolically, $f(x) = x^2 - 2$ *and* $g(x) = \pm\sqrt{x+2}$*, which is not a function. The composition* $g \circ f(x)$ *reduces to:*

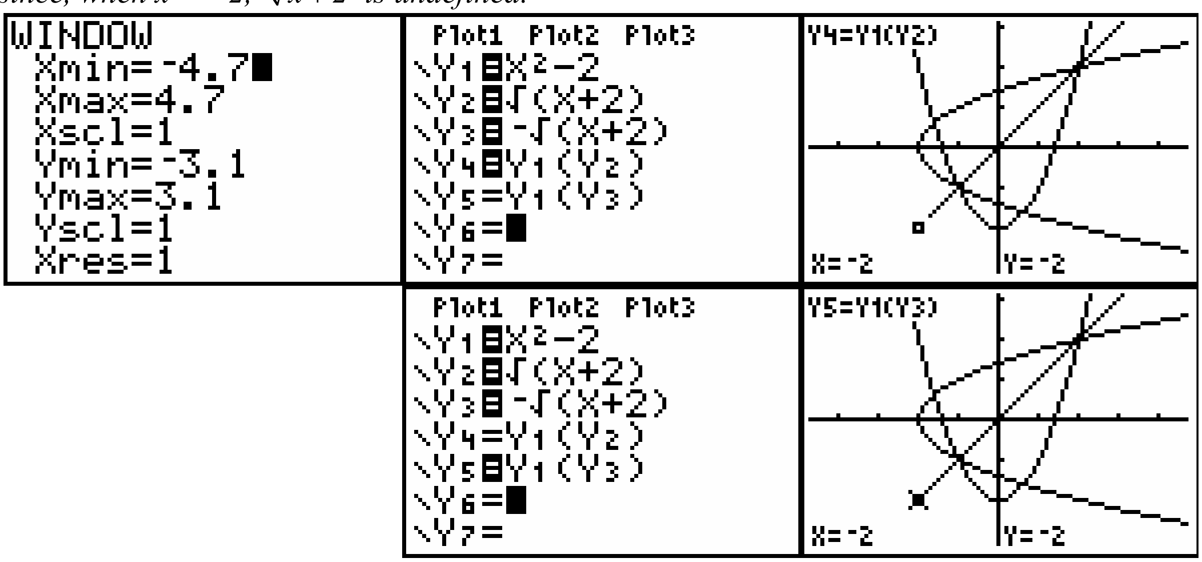
$$\begin{aligned} g \circ f(x) &= \pm\sqrt{\left(x^2 - 2\right) + 2} \\ &= \pm\sqrt{x^2} \\ &= \pm|x| \end{aligned}$$

Case 2: Inverse is not a function – Composition of Original(Inverse)

Numerically, both compositions $Y1 \circ Y2$ *and* $Y1 \circ Y3$ *yield the original x-values only for the domain where* $x \geq -2$ *since, when* $x < -2$, $\sqrt{x+2}$ *is undefined.*



Graphically, both compositions yield the line y = x with the domain restriction of $x \geq -2$ *since, when* $x < -2$, $\sqrt{x+2}$ *is undefined.*



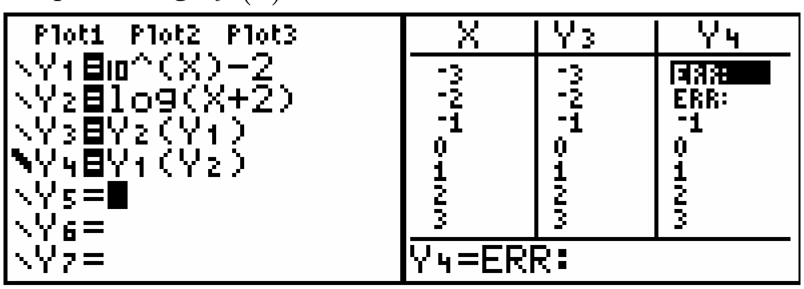
Symbolically, $f(x) = x^2 - 2$ *and* $g(x) = \pm\sqrt{x+2}$*, which is not a function. The composition* $f \circ g(x)$ *reduces to y = x as shown. However, due to range restrictions on the original square root functions (recall that the range of the first function becomes the domain of the second function), there is a limited domain to be operated on by f(x), resulting in a domain restriction on the composed function* $f \circ g(x)$.

$$\begin{aligned} f \circ g(x) &= \left(\pm\sqrt{x+2}\right)^2 - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

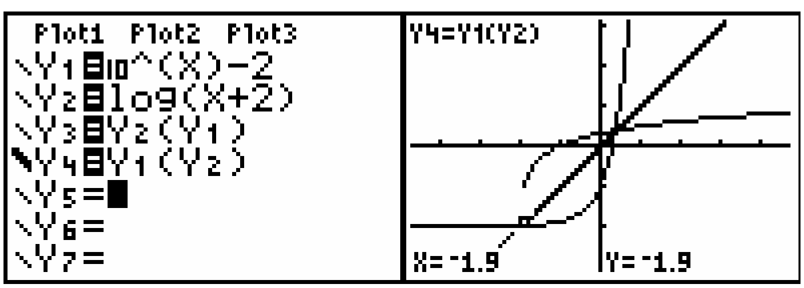
2. **Based on your experiences with linear and quadratic functions, what would you expect to be true about compositions of other types of functions, such as exponential, rational, or polynomial functions? Give examples or counterexamples.**
   Responses may vary. Participants may conjecture that for polynomial functions, patterns exist among even and odd functions. Or, participants may conjecture that since exponential/logarithmic functions exhibit domain restrictions, similar patterns of restrictions on compositions hold true.

   For example, consider the function $f(x) = 10^x - 2$ *and its inverse* $g(x) = \log(x+2)$.

   Numerically, the range of the composition $f \circ g(x)$ *is a subset of the range of the composition* $g \circ f(x)$.



   Graphically, the graph of $f \circ g(x)$ *(bold line shown in the graph) overlaps the graph of* $g \circ f(x)$ *only when x > −2. Hence, the range of* $f \circ g(x)$ *is a subset of the range of* $g \circ f(x)$.



Symbolically: $f(x) = 10^x - 2$ *and* $g(x) = \log(x+2)$*, so:*

$$\begin{aligned} g \circ f(x) &= \log\left(\left(10^x - 2\right) + 2\right) \\ &= \log\left(10^x\right) \\ &= x \log 10 \\ &= x \end{aligned}$$

$$\begin{aligned} f \circ g(x) &= 10^{\log(x+2)} - 2 \\ &= x + 2 - 2 \\ &= x^* \end{aligned}$$

** Since the domain values for g(x) are restricted to x > −2, the range of* $f \circ g(x)$ *is also restricted to x > −2.*

Note to Leader: Ask participant volunteers to share out their examples and counterexamples. Discuss similarities and differences.

Explain

Leaders' Note: *The Maximizing Algebra II Potential (MAP) professional development is intended to be an extension of the ideas introduced in Mathematics TEKS Connections (MTC). Throughout the professional development experience, we allude to components of MTC such as the Processing Framework Model, the emphasis of making connections among representations, and the links between conceptual understanding and procedural fluency.*

Debriefing the Experience:

1. **What concepts did we explore in the previous set of activities? How were they connected?**
   Responses may vary. Participants should observe that inverses are relations whose domains and ranges are reversed. Composition of functions is a foundations for functions concept in that the range of one function is used as the domain for another.

2. **What procedures did we use to describe inverses of functions? How are they related?**
   Tabular, graphical, and symbolic procedures were all used throughout the Explore phase. Ultimately, they are all connected through the numerical relationships used to generate them.

3. **What knowledge from Algebra I do students bring about linear, quadratic, and exponential functions?**
   According to the Algebra I TEKS, students study linear functions in depth. In Algebra I, students transform quadratic functions vertically (stretch/compress, reflect vertically, and translate), analyze the graphs of quadratic functions, solve quadratic equations, and connect roots/zeroes/x-intercepts. In Algebra I, students model growth and decay using exponential functions.

4. **After working with inverses of relations and functions in Algebra II, what are students' next steps in Precalculus or other higher mathematics courses?**
   According to the Precalculus TEKS, students will expand their catalog of curves to include natural logarithms, power functions, and trigonometric functions. Also, students will perform compositions on functions and find inverses of functions, including trigonometric functions such as y = arcsin(x).

Anchoring the Experience:

5. **Distribute to each table group a poster-size copy of the Processing Framework Model.**

6. **Ask each group to respond to the question:**

   Where in the processing framework would you locate the different activities from the Explore phase?

7. **Participants can use one color of sticky notes to record their responses. In future Explain phases, participants will use other colors to record their responses.**

Horizontal Connections within the TEKS

8. **Direct the participants' attention to the second layer in the Processing Framework Model: Horizontal Connections among Strands.**

9. **Prompt the participants to study the Algebra II TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

10. **Invite each table group to share 2 connections that they found and record them so that they are visible to the entire group.**

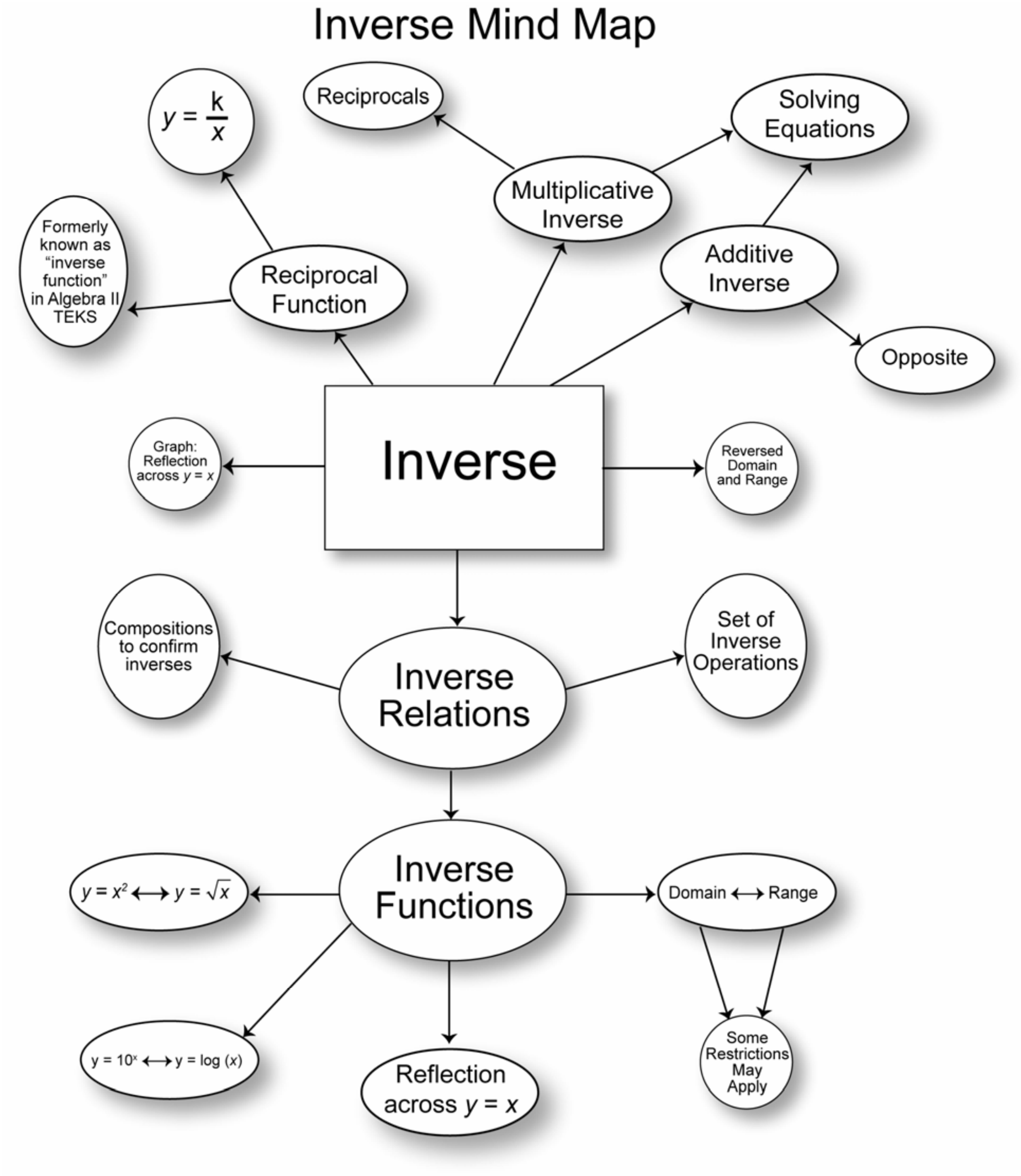
Vertical Connections within the TEKS

11. **Direct the participants' attention to the third layer in the Processing Framework Model: Vertical Connections across Grade Levels.**

12. **Prompt the participants to study the Algebra I, Geometry, Math Models, and Precalculus TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

13. **Invite each table group to share 2 connections that they found, recording so that the entire large group may see.**

14. **Provide each group of participants with a clean sheet of chart paper. Ask them to create a "mind map" for the mathematical term of "inverse."**
    Note to Leader: *If participants ask, be purposefully vague in not clarifying what you mean by "inverse." Part of the purpose of this activity is for participants to realize that students (especially English Language Learners, or ELLs) can be confused by the many meanings of the same term.*


Inverse Mind Map
$y = \frac{k}{x}$
Reciprocals
Solving Equations
Formerly known as "inverse function" in Algebra II TEKS
Multiplicative Inverse
Additive Inverse
Reciprocal Function
Opposite
Inverse
Graph: Reflection across $y = x$
Reversed Domain and Range
Compositions to confirm inverses
Set of Inverse Operations
Inverse Relations
Inverse Functions
$y = x^2 \longleftrightarrow y = \sqrt{x}$
Domain $\longleftrightarrow$ Range
$y = 10^x \longleftrightarrow y = \log(x)$
Reflection across $y = x$
Some Restrictions May Apply

Provide an opportunity for each group to share their mind maps with the larger group. Discuss similarities, differences, and key points brought forth by participants.

Facilitation Questions

- How do the different meanings of the term “inverse” relate to each other?
  Responses may vary. “Inverse” relations use sets of “inverse” operations. “Inverse” trigonometric functions (for example, arcsine) are inverses of the set of trigonometric functions.

- Where would compositions fit into your map?
  Responses may vary depending on participants’ mind maps.

15. **Distribute the Vocabulary Organizer template to each participant. Ask participants to construct a vocabulary model for the term “inverse relations/functions.”**
    Possible response:

My definition

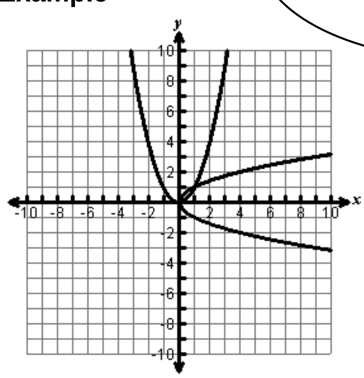
Relations whose domains and ranges are reversed.

Personal Association

Cell phone address book—When I dial, the names are the input and phone numbers are the output. But for incoming calls, the phone number is the input and the name is the output.

Inverse Relations and Functions

Example



Non-Example

$y = \frac{2}{3}x - 5$ *and* $y = \frac{3}{2}x + 5$ *are not inverse functions.*

16. **When participants have completed their vocabulary models, ask participants to identify strategies from their experiences so far in the professional development that could be used to support students who typically struggle with Algebra II topics.**

    Note to Leader: *You may wish to have each small group brainstorm a few ideas first, then share their ideas with the large group while you record their responses on a transparency or chart paper.*

17. How would this lesson maximize performance in Algebra II for teaching and learning the mathematical concepts and procedures associated with inverses of relations and functions?

Responses may vary. Anchoring procedures within a conceptual framework helps students understand what they are doing so that they become more fluent with the procedures required to accomplish their tasks. Problems present themselves in a variety of representations; providing students with multiple procedures to solve a given problem empowers students to solve the problem more easily.

Elaborate

Leaders' Note: *In this phase, participants will extend their learning experiences to their classroom.*

1. **Provide each participant with a copy of the 5E Student Lesson planning template. Ask participants to think back to their experiences in the Explore phase. Pose the following task:**

   What might a student-ready 5E lesson on inverses of relations/functions look like?
   - **What would the Engage look like?**
   - **Which experiences/activities would students explore firsthand?**
   - **How would students formalize and generalize their learning?**
   - **What would the Elaborate look like?**
   - **How would we evaluate student understanding of inverses of relations/functions?**

2. **After participants have recorded their thoughts, direct them to the student lesson for inverses of relations/functions. Allow time for participants to review lessons.**

3. **How does this 5E lesson compare to your vision of a student-centered 5E lesson?**
   Responses may vary.

4. **How does this lesson help remove obstacles that typically keep students from being successful in Algebra II?**
   By connecting symbolic manipulation to conceptual understanding as revealed in other representations (such as graphing), students have other tools with which to solve meaningful problems.

5. **How does this lesson maximize your instructional time and effort in teaching Algebra II?**
   Taking time to create a solid conceptual foundation reduces the need for re-teaching time and effort.

6. **How does this lesson maximize student learning in Algebra II?**
   Using multiple representations and foundations for functions concepts allows students to make connections among different ideas. These connections allow students to apply more quickly and readily their learning to new situations.

7. **How does this lesson accelerate student learning and increase the efficiency of learning?**
   Foundations for functions concepts such as function transformations transcend all kinds of functions. A basic toolkit for students to use when working with functions allows students to rethink what they know about linear and quadratic functions while they are learning concepts and procedures associated with other function families.

8. Read through the suggested strategies on Strategies that Support English Language Learners. Consider the possible strategies designed to increase the achievement of English language learners.

As participants read through the strategies that support English language learners and strategies that support students with special needs, they may notice that eight of the ten strategies are the same. The intention is to underscore effective teaching practices for all students. However, English language learners have needs specific to language that students with special needs may or may not have. The two strategies that are unique to the English language learners reflect an emphasis on language. Students with special needs may have prescribed modifications and accommodations that address materials and feedback. Students with special needs often benefit from progress monitoring with direct feedback and adaptation of materials for structure and/or pacing. A system of quick response is an intentional plan to gather data about a student's progress to determine whether or not the modification and (or) accommodation are (is) having the desired effect. The intention of the strategies is to provide access to rigorous mathematics and support students as they learn rigorous mathematics.

9. What evidence of these strategies do you find in this portion of the professional development?

Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the expected teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening, contributing to the teacher's ability to create an emotionally safe environment for learning.*

10. Which strategies require adaptation of the materials in this portion of the professional development?

Responses may vary. Most of the strategies are incorporated throughout the materials.

11. Read through the suggested strategies on Strategies that Support Students with Special Needs. Consider the possible strategies designed to increase the achievement of students with special needs.

12. What evidence of these strategies do you find in this portion of the professional development?

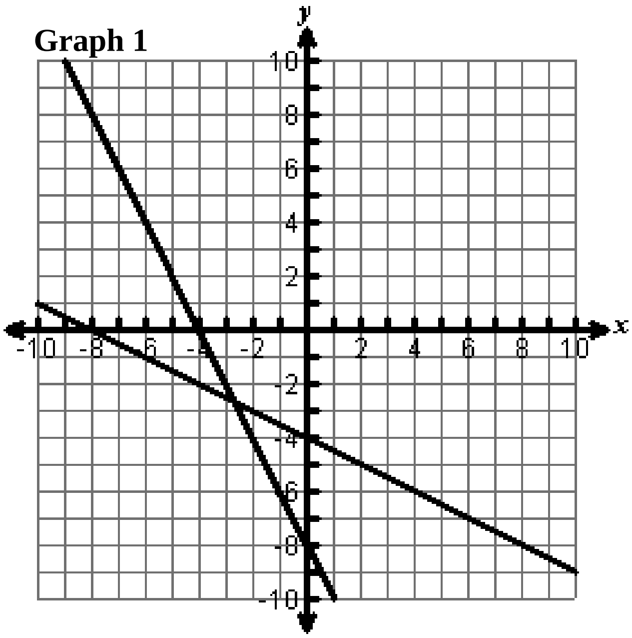
Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening, contributing to the teacher's ability to create an emotionally safe environment for learning.*

13. Which strategies require adaptation of the materials in this portion of the professional development?

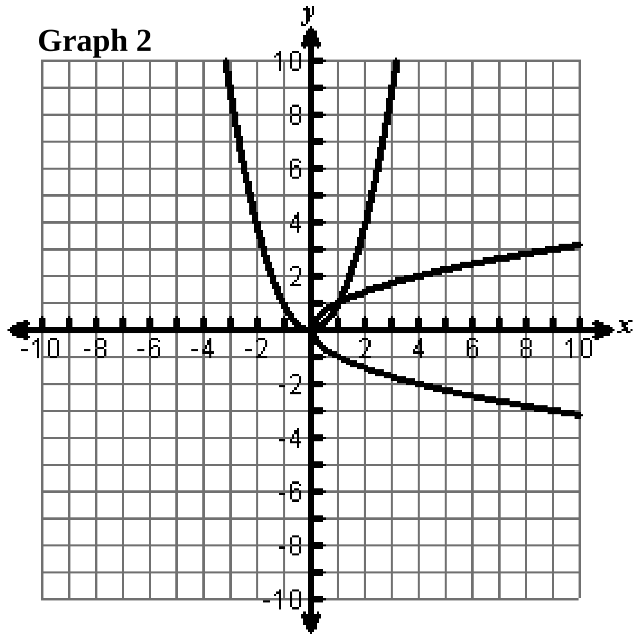
Responses may vary. Most of the strategies are incorporated throughout the materials. Some materials may need to be modified for format.

Know When to Fold 'Em

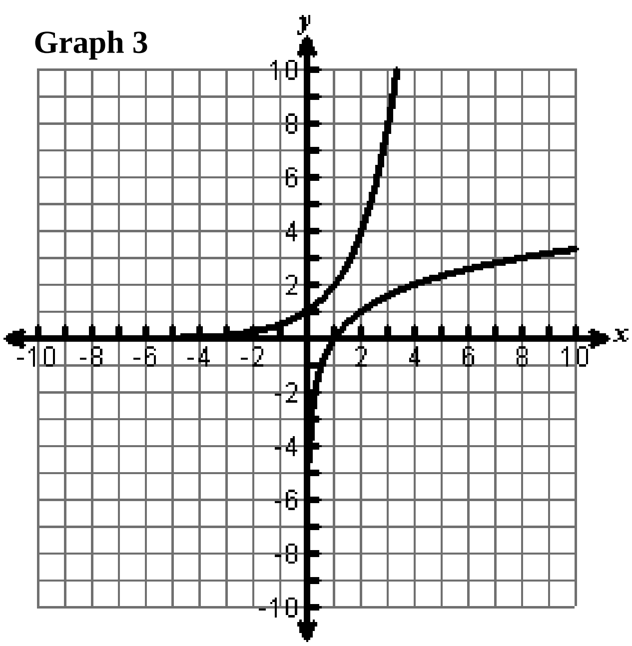
Graph 1



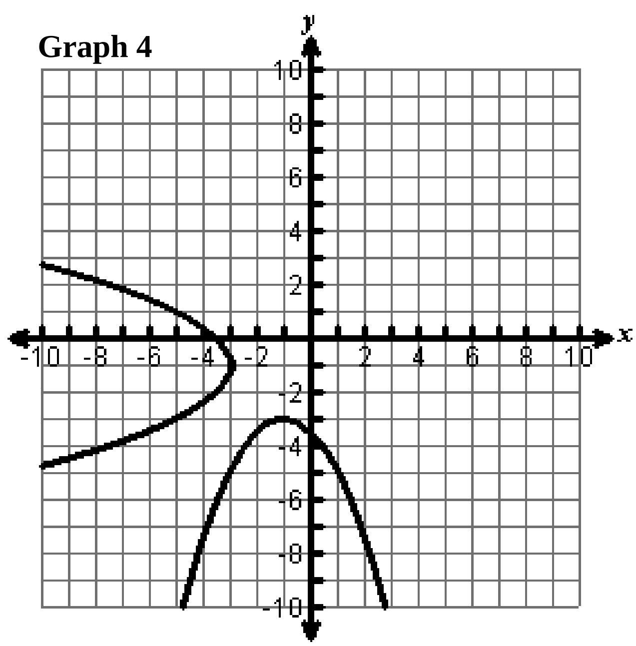
Graph 2



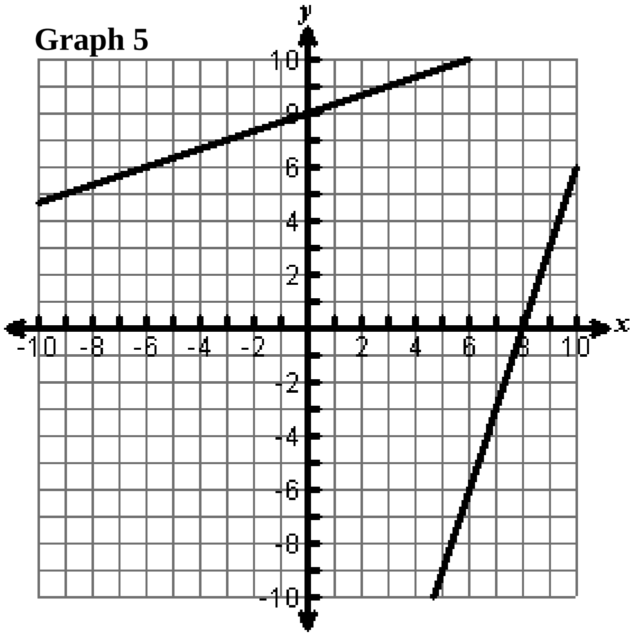
Graph 3



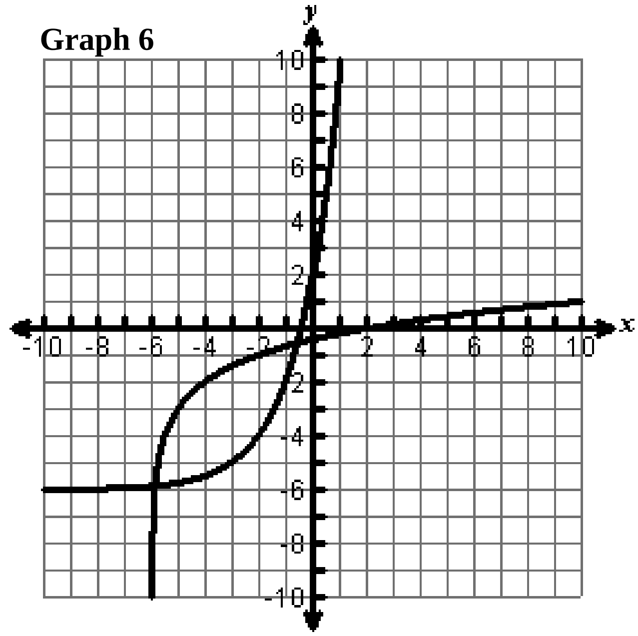
Graph 4



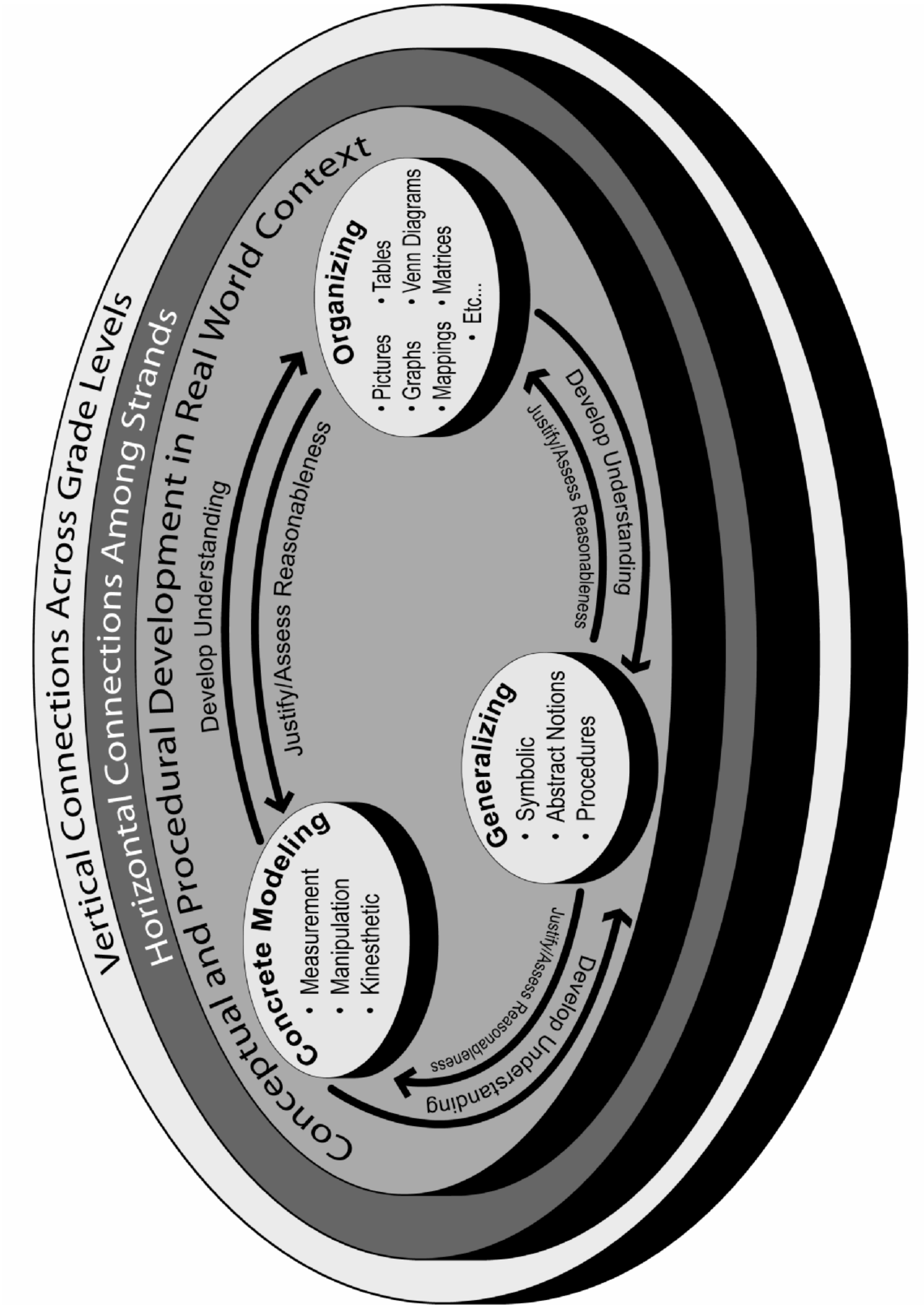
Graph 5



Graph 6



Processing Framework Model



Vocabulary Organizer

My definition	**Personal Association**
Example	**Non-Example**

5E Student Lesson Planning Template

Description	Activity
Engage The activity should be designed to generate student interest in a problem situation and to make connections to prior knowledge. The instructor initiates this stage by asking meaningful questions, posing a problem to be solved, or by showing something intriguing.	
Explore The activity should provide students with an opportunity to become actively involved with the key concepts of the lesson through a guided exploration requiring them to probe, inquire, and question. The instructor actively monitors students as they interact with each other and the activity.	
Explain Students collaboratively begin to sequence events/facts from the investigation and communicate these findings to each other and the instructor. The instructor, acting in a facilitation role, formalizes student findings by providing further explanations and additional meaning or information, such as correct terminology.	
Elaborate Students extend, expand, or apply what they have learned in the first three stages and connect this knowledge with prior learning to deepen understanding. Instructors can use the Elaborate stage to verify students' understandings.	
Evaluate Evaluation occurs throughout students' learning experiences. More formal evaluation can be conducted at this stage. Instructors can determine whether the learner has reached the desired level of understanding the key ideas and concepts.	

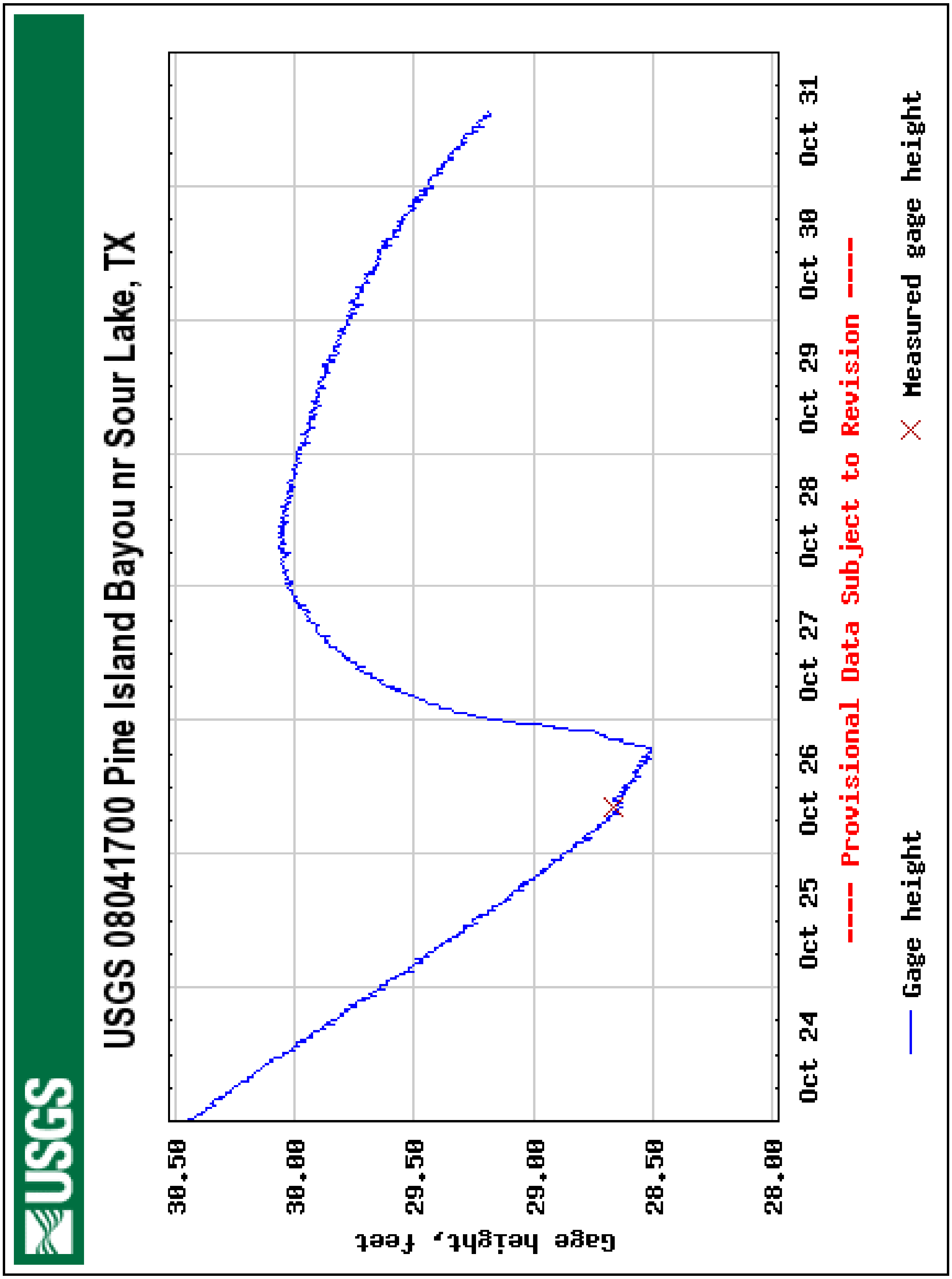
Strategies that Support English Language Learners (ELL)

Strategy	Explore, Explain, Elaborate 1
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond.	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Be conscious of tone and diction. Speak slowly and distinctly.	
Incorporate language skills (reading, writing, speaking, and listening) into instruction.	

Strategies that Support Students with Special Needs

Strategy	Explore, Explain, Elaborate 1
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond.	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Use a system of quick response to needs and accommodations including progress monitoring to inform instruction.	
Accommodate materials for format, structure, sequence, etc. as needed.	

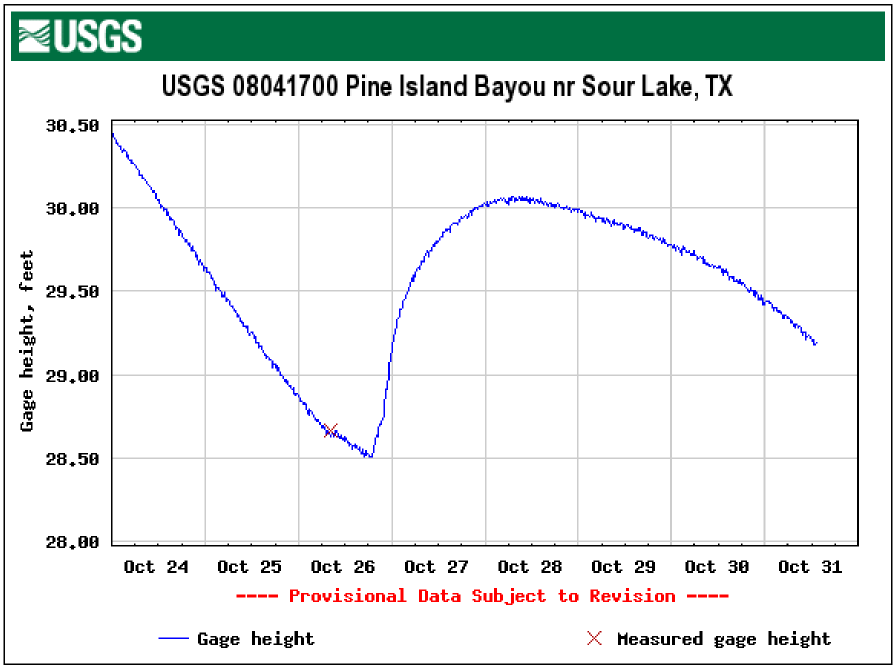
Transparency: Pine Island Bayou



Participant Pages: Inverses of Functions

<u>Part 1</u>: Generating an Inverse Relation

During the Engage phase, you examined graphs of streams' gage heights versus time. Sometimes, hydrologists are concerned about a particular gage height and when the gage for the stream measured that height. In those cases, it makes sense to consider a plot of time versus gage height.



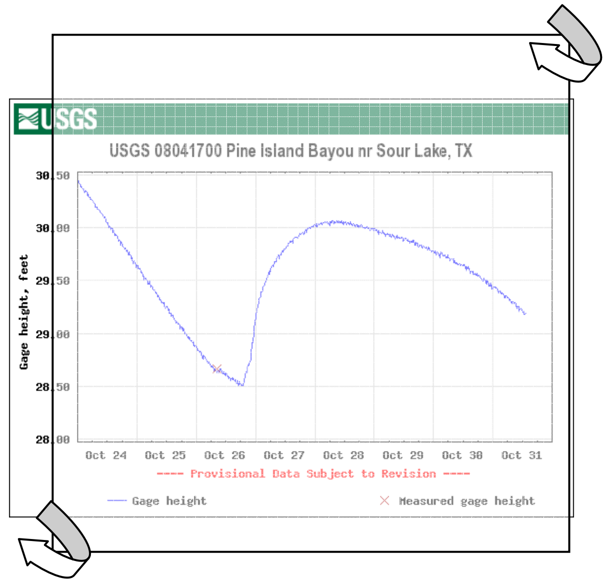
1. For the scatterplot of gage height versus time, what are the inputs and outputs?

2. Does this scatterplot represent a functional relationship? How can you tell?

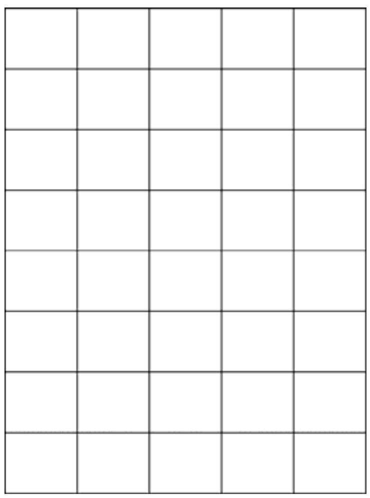
3. What would the graph of the same data look like if we plotted time versus gage height? Sketch a prediction of the graph.

Trace the graph of gage height versus time onto a sheet of patty paper. Be sure to trace the plot and the axes then label your axes on your patty paper.

Take the top-right corner in your right hand and the bottom-left corner in your left hand and flip the patty paper over. Align the origin on the patty paper with the origin of your original graph and align the axes on the patty paper with the axes on the graph.



4. Sketch the resulting graph.



5. How is the new graph similar to the original graph? How are they different?

6. Is the new graph a function? How do you know? Given the situation, does it make sense for the graph to be or not to be a function? Explain your reasoning.

On the activity sheet, **Know When to Fold 'Em**, trace each graph (both curves) onto a piece of patty paper. Be sure to also trace and label the axes.

Fold the patty paper to find all possible lines of symmetry for each pair of graphs shown. Identify the equation for each possible line of symmetry. Record the number of lines of symmetry and their equations for each graph on your recording sheet.

	Total Number of Lines of Symmetry	**Equation(s) of line(s) of symmetry**
Graph 1		
Graph 2		
Graph 3		
Graph 4		
Graph 5		
Graph 6		

7. What patterns do you notice among the lines of symmetry for each of the graphs?

8. Which transformation describes the folds across a line of symmetry for your graphs?

9. Make a summary statement describing the relationship between each pair of curves in the set of given graphs.

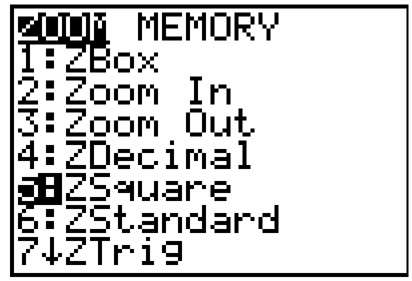
<u>Part 2</u>: Numerical and Symbolic Representations

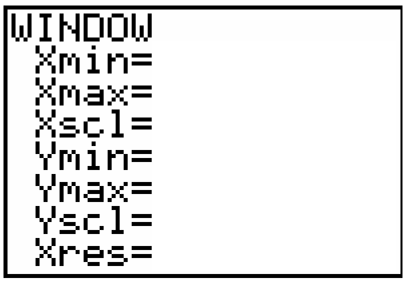
The following coordinates were used to create a geometric design for a quilting pattern.

List 1	1	1	0	0	−1	−1	−2	−1	−2	−1	1
List 2	2	4	5	4	5	4	4	3	3	2	2

1. Create a connected scatterplot of L_2 versus L_1. Sketch your graph and describe your viewing window.

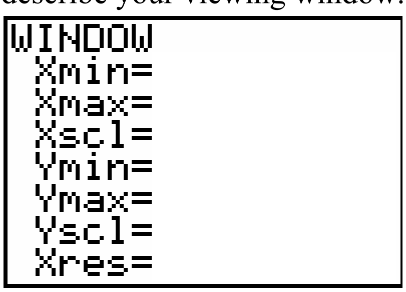
   TIP: Be sure to use a square window. After selecting your appropriate domain and range, use the Zoom-Square feature to square out the grid in your viewing window.





2. What do you think would happen to the graph if we reversed the *x*- and *y*-values?

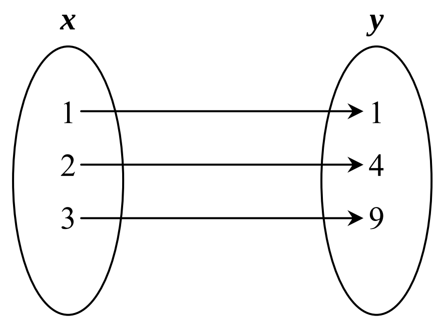
3. Create a second connected scatterplot of L_1 versus L_2. Use a different plot symbol for this scatterplot. Graph this scatterplot with your original scatterplot. Sketch your graph and describe your viewing window. You may need to re-square your viewing window.



4. Compare the two scatterplots. How are they alike? How are they different?

5. How are the x- and y-coordinates related from the first scatterplot to the second scatterplot? How could you represent this relationship symbolically?

Recall that a mapping shows how domain elements for a relation relate, or "map to" their corresponding range elements. For example, the following mapping shows how the x-values $\{1, 2, 3\}$ map to their corresponding y-values $\{1, 4, 9\}$ for the function $y = x^2$.



Enter the functions $Y_1 = 2x - 8$ and $Y_2 = \frac{1}{2}x + 4$ into your graphing calculator.

6. Use the table feature of your graphing calculator to generate values for a mapping for Y1 to show the replacement set for y when $x = \{5, 6, 7, 8, 9\}$.

7. Generate a mapping for Y2 to show the replacement set for y when $x = \{2, 4, 6, 8, 10\}$.

8. How are the two mappings related?

9. In each mapping, to how many y-values does any given x-value map? Would you expect this to be true for other domain and range elements? How do you know?

10. What does this reveal about the relationships in each mapping?

11. In each mapping, how many x-values map to any given y-value? Would you expect this to be true for other domain and range elements? How do you know?

12. What does this reveal about the relationships in each mapping?

13. Examine the graphs of Y1 and Y2. How are they related? (Hint: Be sure you are using a square viewing window.)

Enter the functions $Y_3 = \frac{2}{3}x - 7$ and $Y_4 = \frac{3}{2}x + 7$ into your graphing calculator.

14. Use the table feature of your graphing calculator to generate a mapping for Y3 to show the replacement set for y when $x = \{0, 3, 6, 9\}$.

15. Use the table feature of your graphing calculator to generate a mapping for Y4 to show the replacement set for y when $x = \{-7, -5, -3, -1\}$.

16. How are the two mappings related?

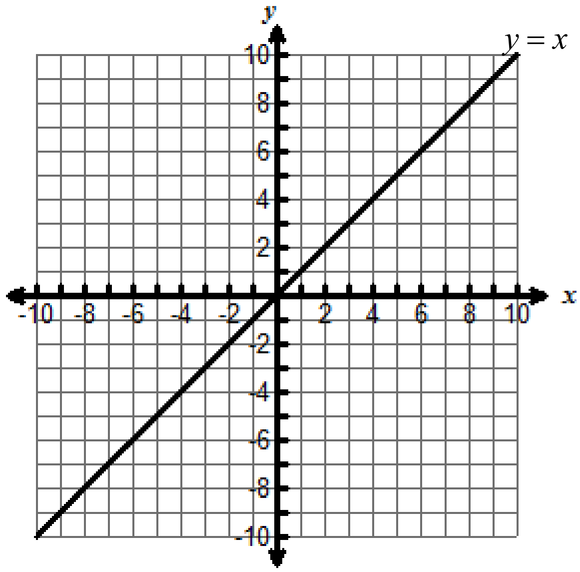
17. Examine the graphs of Y3 and Y4. How are they related? (Hint: Be sure you are using a square viewing window.)

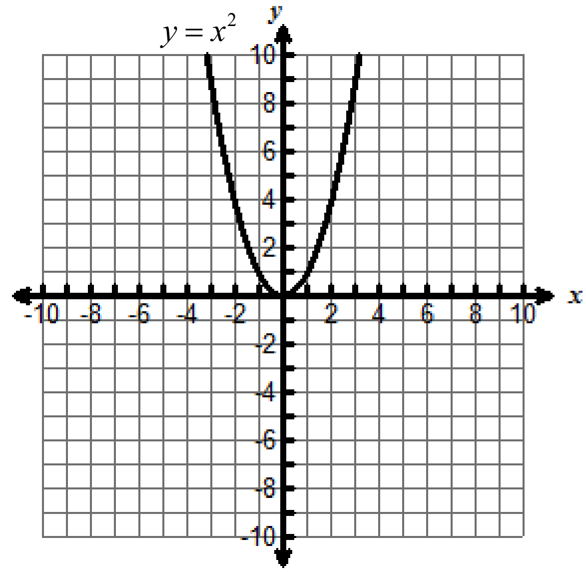
18. The functions in Y1 and Y2 are called "inverse relations" whereas the functions in Y3 and Y4 are not. Based on your mappings, graphs, and equations, why might this be the case?

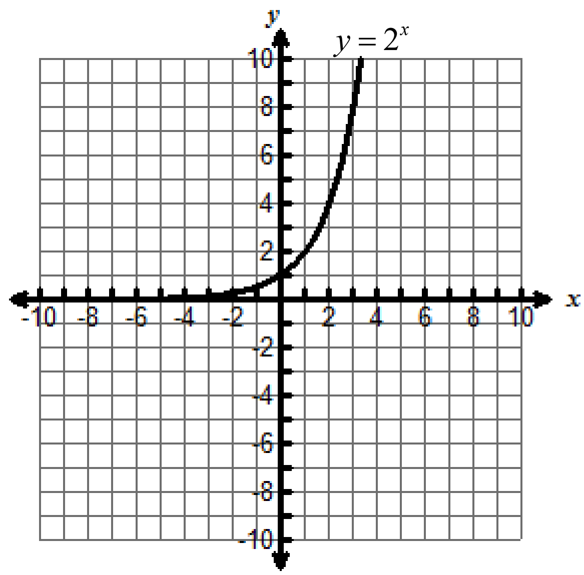
19. Based on your response to the previous question, how might we describe inverse relations graphically, numerically, and symbolically?

Part 3: Investigating Linear Functions

In Algebra 1, students investigate linear, quadratic, and exponential functions. The graphs of the parent functions are shown.

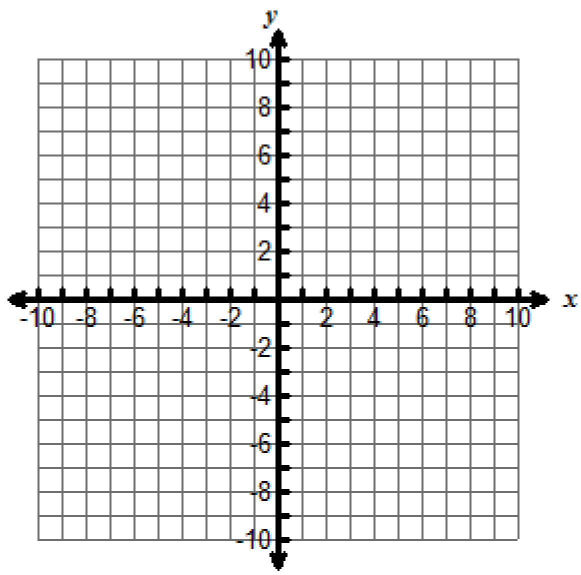






Trace each parent function onto a separate piece of patty paper. Be sure to trace and label the axes as well.

1. Reflect the linear parent function across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of the inverse of the linear parent function? How do they compare with the original function?

3. Is the inverse of the linear parent function also a function? How do you know?

4. What kind of function is the inverse of a linear function?

5. What is the inverse of the function $y = \frac{2}{5}x - 7$? Find the inverse using at least two different methods.

6. How did you determine the inverse?

7. What concepts and procedures did you apply to determine the inverse?

8. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{2}{5}x - 7$?

9. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{2}{5}x - 7$?

10. How do these two sets of operations compare?

11. Describe the graph of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent functions.

12. Describe the graph of the inverse of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent functions.

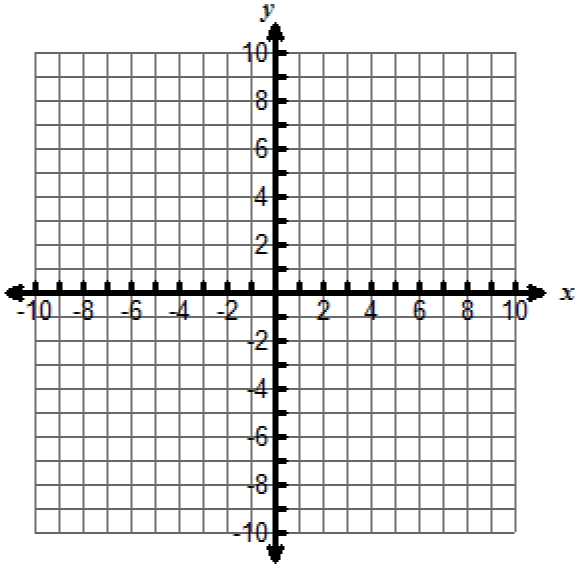
13. Compare the graphs of $y = \frac{2}{5}x - 7$, its inverse, and the line $y = x$.
    a. What do you notice about the intercepts of the graphs?

    b. Where are the three graphs concurrent? What is the significance of this point?

    c. In terms of transformations, how do the graphs of the original function and its inverse compare?

<u>Part 4</u>: Quadratic Functions

1. Reflect the quadratic parent function across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of the inverse of the quadratic parent function? How do they compare with the original function?

3. Is the inverse of the quadratic parent function also a function? How do you know?

4. If the inverse is not a function, how can we restrict the domain and/or range of the original function so that the inverse is also a function?

5. What kind of function is the inverse (range restricted)of a quadratic function?

6. Is either the original parent function or its range restricted inverse one-to-one? How do you know?

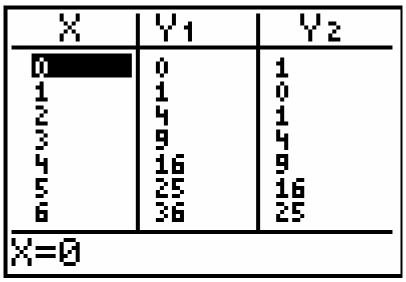
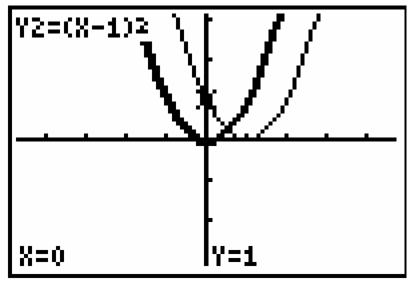
7. What is the inverse function of the function $y = -3(x-1)^2 + 2$? Find the inverse using at least two different methods.

8. How did you determine the inverse?

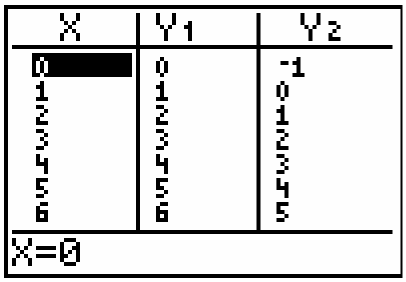
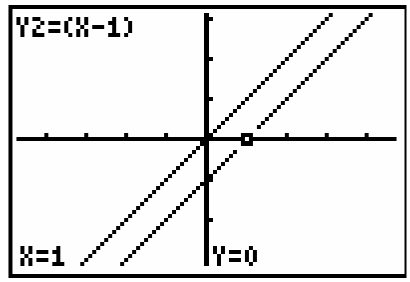
9. What concepts and procedures did you apply to determine the inverse?

10. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = -3(x-1)^2 + 2$?

11. From the perspectives of number operations and transformations, develop the quadratic function, $y = -3(x-1)^2 + 2$, from the parent function $y = x^2$. Include tabular, graphical, and symbolic representations of the number operation as it applies to the function.

Number Operation	**Tabular**	**Graphical**	**Symbolic**
Subtract 1 from the x-value in the parent function $y = x^2$.			$y = (x-1)^2$

12. Now, use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the quadratic function, $y = -3(x-1)^2 + 2$, from $y = x$.

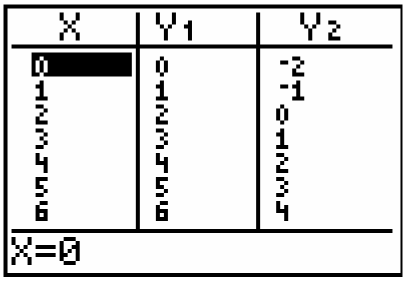
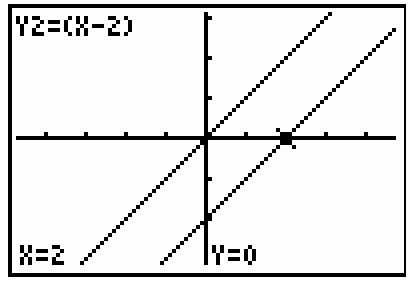
<table>
<tr><th>Number Operation</th><th>Tabular</th><th>Graphical</th><th>Symbolic</th></tr>
<tr>
<td>Subtract 1 from the parent function $y = x$</td>
<td>
<table>
<tr><th>X</th><th>Y1</th><th>Y2</th></tr>
<tr><td>0</td><td>0</td><td>-1</td></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
<tr><td>2</td><td>2</td><td>1</td></tr>
<tr><td>3</td><td>3</td><td>2</td></tr>
<tr><td>4</td><td>4</td><td>3</td></tr>
<tr><td>5</td><td>5</td><td>4</td></tr>
<tr><td>6</td><td>6</td><td>5</td></tr>
</table>
X=0</td>
<td>
</td>
<td>$y = x - 1$</td>
</tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
</table>

13. How does each successive number operation transform the function numerically, graphically, and symbolically?

14. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = -3(x-1)^2 + 2$?

15. How does this set of operations compare to the operations applied to generate the function $y = -3(x-1)^2 + 2$?

16. Use number operations to describe what happens to the domain elements, represented by the variable x, as you develop the square root inverse of the function, $y = -3(x-1)^2 + 2$, from $y = x$.

Number Operation	**Tabular**	**Graphical**	**Symbolic**
Subtract 2 from the parent function *y* = *x*.			$y = x - 2$

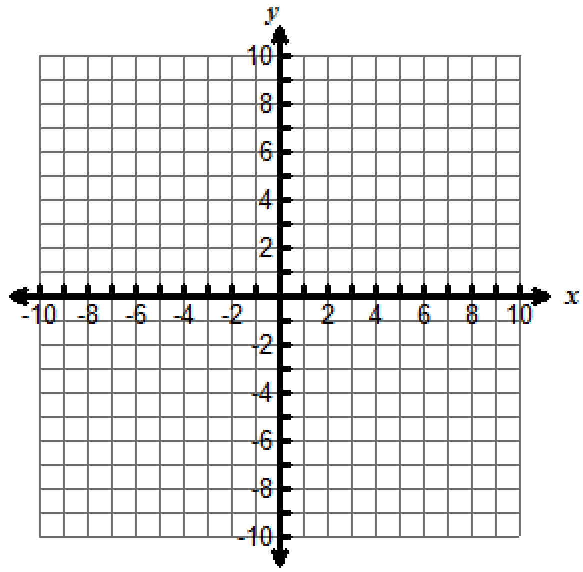
17. How do the inverse operations relate to the inverse function?

18. Generalize how inverse relations for quadratic functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.

Number Operation	Tabular	Graphical	Symbolic

<u>Part 5:</u> Exponential Functions

1. Reflect the exponential parent function, $y = 2^x$, across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of $y = 2^x$?

3. What is the domain and range of the inverse of $y = 2^x$? How do they compare with the original function?

4. What asymptote(s) does the original function, $y = 2^x$ have? Why does this asymptote exist?

5. What asymptote(s) does the inverse of $y = 2^x$ have? How do they compare to the asymptotes of the original function?

6. Is the inverse of $y = 2^x$ also a function? How do you know?

7. What kind of function is the inverse of an exponential function?

8. Is either the original parent function or its inverse one-to-one? How do you know?

9. What is the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$? Find the inverse using at least two different methods.

10. How did you determine the inverse?

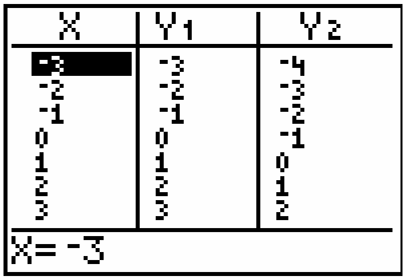
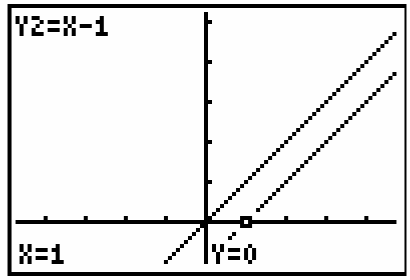
11. What concepts and procedures did you apply to determine the inverse?

12. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{1}{2}(10)^{x-1} + 3$?

13. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$?

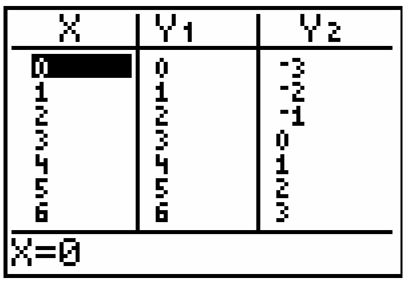
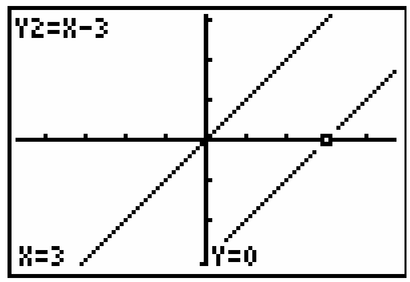
14. How do these two sets of operations compare?

15. Use number operations to describe what happens to the domain elements, represented by the variable x, as you develop the exponential function, $y = \frac{1}{2}(10)^{x-1} + 3$, from $y = x$.

Number Operation	**Tabular**	**Graphical**	**Symbolic**
Subtract 1 from the parent function $y = x$			$y = x - 1$

16. How does each successive number operation transform the function numerically, graphically, and symbolically?

17. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the logarithm inverse of the function, $y = \frac{1}{2}(10)^{x-1} + 3$, from $y = x$.

<table>
<tr><th>Number Operation</th><th>Tabular</th><th>Graphical</th><th>Symbolic</th></tr>
<tr><td>Subtract 3 from the parent function $y = x$</td><td>
</td><td>
</td><td>$y = x - 3$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
</table>

18. How do the inverse operations relate to the inverse function?

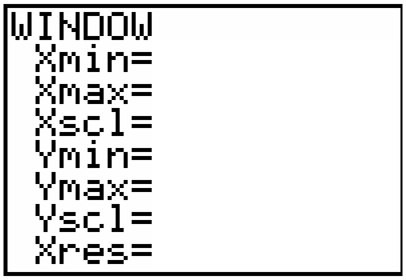
19. Generalize how inverse relations for exponential functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.

Number Operation	Tabular	Graphical	Symbolic

<u>Part 6</u>: Compositions of Functions

Enter the function $y = 2x - 1$ into *Y1* of your graphing calculator's function editor. Enter the inverse of this function into *Y2*. Enter the composition of *Y2* and *Y1* into *Y3*.

1. Sketch the graphs of the three functions (describe your viewing window). What relationships and patterns do you notice?



2. Look at the table values for each of the three functions. What do you notice?

3. How could you represent the composition of these two functions with a mapping?

4. What effect does composition of inverse functions have? Why do you think this is so?

5. Does the order of composition matter? Explain your answer.

6. Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse function. Describe the result numerically, graphically, and symbolically.

<u>Part 7</u>: Extension

1. Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse <u>relation</u>. Describe the result numerically, graphically, and symbolically.
   Case 1: Inverse is not a function – Composition of Inverse(Original)

   Case 2: Inverse is not a function – Composition of Original(Inverse)

2. Based on your experiences with linear and quadratic functions, what would you expect to be true about compositions of other types of functions, such as exponential, rational, or polynomial functions? Give examples or counterexamples.