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maximizing Algebra II performance

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LITERATURE REVIEW

“Becoming mathematically proficient is necessary and appropriate for all students” (National Research Council, 2001, p.142). Mathematical proficiency is expected of all students enrolled in the public school system in the state of Texas.

What do we know about the students who are not meeting the state standards for mathematical proficiency? As described in Table 1, Table 2, and Table 3, the student groups with the greater percentages of students failing to demonstrate mathematical proficiency include at-risk learners and English language learners (ELL) in the ESL, Limited English Proficient, and Bilingual categories.

Table 1. *Percentages of Grade 9 Students Failing to Meet Panel Recommendation*

	2003	**2004**	**2005**	**2006**
At-Risk	82	79	72	70
Economically Disadvantaged	72	65	58	58
Special Education	86	80	73	74
ESL	89	86	83	81
Limited English Proficient	89	86	82	81
Bilingual	77	65	59	59

From Texas Education Agency, http://www.tea.state.tx.us/student.assessment/reporting/results/summary/sum06

Table 2. *Percentages of Grade 10 Students Failing to Meet Panel Recommendation*

	2003	**2004**	**2005**	**2006**
At-Risk	79	77	77	67
Economically Disadvantaged	68	64	57	53
Special Education	85	81	74	72
ESL	83	83	83	77
Limited English Proficient	83	82	82	77
Bilingual	70	58	56	54

From Texas Education Agency, http://www.tea.state.tx.us/student.assessment/reporting/results/summary/sum06

Table 3. *Percentages of Grade 11 Students Failing to Meet Panel Recommendation*

	2003	**2004**	**2005**	**2006**
At-Risk	82	55	48	36
Economically Disadvantaged	72	47	42	37
Special Education	88	69	62	54
ESL	86	67	66	58
Limited English Proficient	85	66	65	57
Bilingual	83	41	53	42

From Texas Education Agency, http://www.tea.state.tx.us/student.assessment/reporting/results/summary/sum06

Research on the teaching and learning that are occurring in effective mathematics programs is summed up by Hiebert (2003):

> One of the most reliable findings from research on teaching and learning is that students learn what they are given opportunities to learn. "Opportunity to learn" is a significant phrase. It means more than just receiving information. Providing an opportunity to learn means setting up the conditions for learning that take into account students' entry knowledge, the nature and purpose of the activities, the kind of engagement required, and so on. …Providing an opportunity to learn what is intended means providing the conditions in which students are likely to *engage* in tasks that involve the relevant content. Such engagement might include listening, talking, writing, and reasoning, and a variety of other intellectual processes.

The teaching component of an effective mathematics program relies on data gained from formative and summative assessment opportunities. (p.10)

To meet the needs of students failing to demonstrate mathematical proficiency, research-based strategies and research-based insights provide direction for addressing the needs of students in at-risk situations, English language learners, and other students who have historically struggled with mathematics. These strategies and insights highlight four primary areas of concern: entry knowledge of the student, nature and purpose of classroom activities, student engagement, and assessment designed to inform instruction.

<u>Entry Knowledge</u>

Entry knowledge is the knowledge and understanding with which a student enters a learning context. Such knowledge is influenced by a student's background, including linguistic and socioeconomic factors (Ball, 1997), as well as prior educational experiences. Teacher awareness of the extent of a student's entry knowledge plays a pivotal role in providing instruction for at-risk learners and English language learners (Ball, 1997). "Teachers must learn what their students know so as to know how to approach a topic, and they must also probe what students are learning from lessons" (Ball, 1997, p. 732). Teachers learn about their students' entry knowledge by listening to what students say without rephrasing what they say (Ball, 1997). As teachers probe student understanding through questioning and instructional conversation and reflect upon what they have heard, the teachers gain a better sense of students' entry knowledge.

Educators can build on entry knowledge through purposeful vocabulary instruction. Marzano (2004) found that the mean scores for students receiving purposeful vocabulary instruction were 0.97 standard deviation greater than the mean scores of students who did not receive purposeful vocabulary instruction. Attributes of such purposeful vocabulary instruction include introducing of new vocabulary through exposure rather than stated definitions; using language-based and imagery-based representations, such as the Verbal and Visual Word Association strategy (Readance, Bean, & Baldwin, 2001) for vocabulary; refining of vocabulary knowledge through multiple exposures to the terminology in contexts; teaching prefixes, roots, and suffixes enhances students' understanding of words; facilitating student discourse related to the vocabulary being learned; and emphasizing those words that contribute most to content-area learning (Marzano, 2004).

Nature and Purpose of Activities

When considering the nature and purpose of activities to support learning, one must consider the cognitive goals required for the learning to take place and the instructional design issues that impact those goals. The goals for mathematical learning include mathematical fluency and problem solving proficiency. Mathematical proficiency is an integration of and interdependence between five key elements:

- Conceptual understanding,
- Procedural fluency,
- Strategic competence,
- Adaptive reasoning, and
- Productive disposition (National Research Council, 2001).

A student demonstrates conceptual understanding by giving evidence of comprehension of mathematical concepts, operations, and relations. Evidence of procedural fluency includes demonstrated skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Computational fluency, the knowledge of basic facts and efficient and accurate methods for performing mathematical computations, is embedded within procedural fluency (NCTM, 2000). These conceptual and procedural elements support strategic competence, the "ability to formulate, represent, and solve mathematical problems" (National Research Council, 2001, p. 5). These elements are supported by adaptive reasoning, the "capacity for logical thought, reflection, explanation, and justification" (National Research Council, 2001, p.5). Intertwined with these elements is a student's productive disposition, his or her "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (National Research Council, 2001, p. 5). Evidence or lack of evidence of each of these elements of mathematical fluency independently, and in concert with each other, provides insight into the extent to which a student demonstrates mathematical fluency.

A second cognitive goal addresses problem solving. "In mathematics, student problem-solving achievement increased with instruction that utilized meaningful, contextualized problems, taught students how to prepare and solve problems systematically, and provided social contexts and peer modeling" (Barley et al., 2002, p. 46). Evidence of mathematical proficiency is demonstrated when students move beyond simple computation to problem-solving contexts.

Instructional Design

Student Needs

Careful consideration must be made when planning for instruction so that each student receives opportunities to develop mathematical fluency and problem solving proficiency. The needs of at-risk learners and English language learners must be addressed when creating instructional designs for learning. There is a broad consensus in studies on at-risk populations on five principles that should guide educational efforts to meet the needs of at-risk learners and English language learners (CREDE, 1997):

- Integrate the efforts of teachers and students in the learning process.
- Embed language and the literacy of instruction in all instructional activities.
- Contextualize teaching and curriculum in the entry knowledge and experiences of the students.

- Challenge students toward cognitive complexity.
- Engage students through discourse, especially instructional conversation.

"Work that is carried out collaboratively for a common objective and the discourse that accompanies the process contribute to the highest level of academic achievement" (CREDE, 1997, p. 2). The discourse should include content-specific vocabulary, questions, problem posing, and representations so that students become literate in communicating about instruction and learning. The process of questioning and sharing thought processes and background knowledge establishes the tone of instructional conversations between students and teachers. As teachers listen carefully to the student, they make conjectures about the student's intended meanings and adjust their responses to assist the student's efforts to learn. To support such discourse, "[a]t-risk learners require instruction that is cognitively challenging, that is, instruction that requires thinking and analysis, not only rote, repetitive, detail-level drills" (CREDE, 1997, p. 4). The goal is a balance between fluency and problem solving. At-risk learners and English language learners have a need to learn and practice the language of mathematical fluency and problem solving.

Classroom Practice

Classroom practice that provides opportunities to develop mathematical fluency and problem-solving proficiency is cognitively oriented. Such practice helps students improve the depth and clarity of their thinking, become independent learners, and become more proficient in successfully completing complex, rigorous academic tasks (Barley et al., 2002). At-risk learners benefit from practices where "[t]eachers are modeling, explaining, prompting, and discussing combinations of metacognitive and cognitive strategies" (Barley et al., 2002, p. 47). English language learners benefit when teachers model, explain, prompt, and discuss strategies through different modalities; through connections between new concepts, entry knowledge, and prior learning; through student-generated refinements and reflections about their own work; and through individualized experiences (Huling & Beck, 2005).

At-risk learners also benefit when they model, explain, prompt, practice, and discuss their thought processes and reflect upon these thought processes (Barley et al., 2002). English language learners benefit when partnered with peers to enhance opportunities to model, explain, prompt, practice, and discuss. To support the use of appropriate vocabulary and language, the students and teachers should make frequent use of models, mind maps, word walls, and key vocabulary. These experiences aid in the learning of math content and the English language while sustaining active participation in the learning experience (Huling & Beck, 2005).

Instructional conversations address the needs of at-risk and English language learners. As a practice, instructional conversations between teachers and students and between students provide opportunities to share preliminary solutions, receiving feedback on content and process. Research suggests that students benefit from this cognitively-based classroom practice (Barley et al., 2002). Students also benefit from conversations that require students to compare and contrast concepts and procedures, articulating similarities and differences (Huling & Beck, 2005 and Marzano, 2001).

Analysis of short-term gain suggests that when these classroom practices reflecting effective, cognitively-oriented instruction are used, students benefit. Students show interest in content and tasks, choose to engage in the learning process, and successfully perform rigorous academic tasks (Barley et al., 2002).

Curriculum

"A 'coherent' curriculum is one that holds together, that makes sense as a whole; and its parts, whatever they are, are unified and connected by that sense of the whole" (Beane, 1995, p. 3). Curriculum that meets the needs of at-risk learners and English language learners considers "the whole" and "the parts" of content and instructional processes in light of these students' needs. Learning is enhanced when a teacher identifies specifically the parts and the types of knowledge that form the focus of the "whole" that constitutes each lesson, each unit, and each content (Marzano, 2003). English language learners benefit from seeing a "cohesive big-picture of units and lessons within units" (Huling & Beck, 2005).

Marzano (2003) also determined that "learning requires engagement in tasks that are structured or sufficiently similar for effective transfer of knowledge" (p. 109). Barley et al. (2002) found that activity materials that include varied texts and problem types that are relevant to the students contribute positively to the learning of at-risk students. Additional supporting structures for at-risk students include strategies for analyzing and preparing to meet the demands of texts and problems, "how-to" solutions, procedures, aids to comprehension (Barley et al., 2002). The curriculum for at-risk students should use structured problem-solving tasks while assisting students in developing strategies to successfully work through and learn from these tasks.

English language learners also benefit from challenging, age-appropriate, and well-paced tasks that incorporate contextually-based problems and problem solving. Concepts should be presented accurately, logically, and in engaging ways that incorporate multiple representations including concrete representations, semi-concrete representations, and abstract representations (Huling & Beck, 2005). Marzano (2003) highlights that "learning requires multiple exposures to and complex interactions with knowledge" (p. 112). These repeated exposures and interactions should include contexts centering on conceptual, computational, and procedural fluency as well as strategic competence, adaptive reasoning, and productive disposition (National Research Council, 2001). These curriculum experiences that reflect the teaching and learning of mathematics are the product of interactions among the teachers, the students, and the mathematics in an instructional triangle as shown in Figure 1.

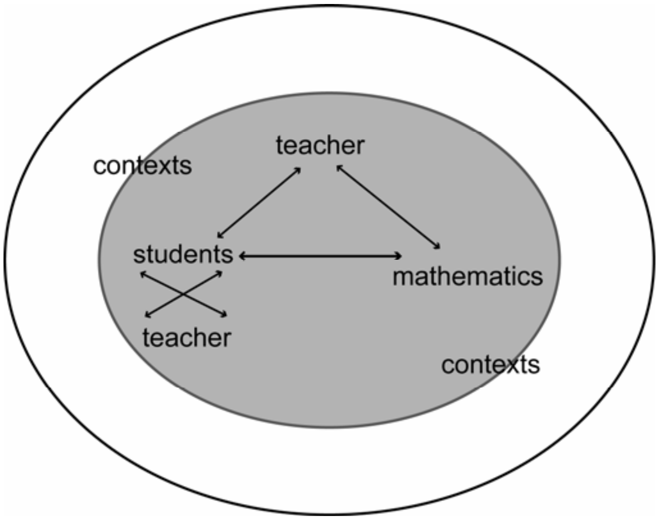

Figure 1. Teaching for Mathematical Proficiency. (National Research Council, 2001, p. 9).

These interactions when well-structured within a curriculum that is focused on appropriate grade-level standards for mathematical proficiency result in the transfer of knowledge described by Marzano (2003). The teaching of such curriculum "promotes learning over time so that the learning yields mathematical proficiency" (National Research Council, 2001, p. 313).

Student Engagement

For at-risk students to engage actively in the content prescribed by mathematics curricula, attention should be given to the developmental, motivational, social, metacognitive, and affective features of instruction (Barley et al., 2002). To address the developmental needs of fifth graders, consideration should be given to the concrete experiences that students need to actively engage in the learning of mathematics content. The National Research Council (2001) found that "[s]imply putting concrete materials on desks or suggesting to students that they might use manipulatives is not enough to guarantee that students will learn appropriate mathematics from them" (p. 353). The manner in which students engage or interact with manipulatives is of tantamount importance.

> Students may not look at these objects the same way adults do, and it can be a challenge for students to see mathematical ideas in them. When students use a manipulative, they need to be helped to see its relevant aspects and to link those aspects to appropriate symbolism and mathematical concepts and operations. Observational studies have documented cases in which students were taught to use manipulatives in a prescribed way to perform "wooden algorithms." If students do not see the connections among object, symbol, language, and idea, using manipulatives becomes just one more thing to learn rather than a process leading to a larger mathematical learning goal (National Research Council, 2001, pp. 353-354).

Manipulatives and other concrete materials provide tools for engaging students at a developmentally appropriate level. However, the planning, activity, and questioning that support the use of manipulatives must be part of "the whole" of the curriculum, part of the thought processes of students, and part of the instructional conversations that take place during mathematics learning.

As students engage with mathematics, they are motivated by feedback on their knowledge gain on conceptual and procedural understandings as well as strategic competence, adaptive reasoning, and productive disposition. Marzano (2003) found that this feedback benefited the learning process.

> When individual growth is the criterion for success, then all students can experience success regardless of their comparative status. To accomplish this, two elements are required: (1) an assessment of the achievement level at which students enter a class or unit of instruction, and (2) an assessment of the achievement level at which students exit the class or unit of instruction (Marzano, 2003, p. 149).

Marzano (2003) also found that students are motivated by tasks that are inherently engaging. Covington (1992) and Marzano (2003) suggest that a student's motivation to learn increases when manageable challenges are part of the curriculum and instruction. These activities arouse curiosity, "providing sufficient complexity so that outcomes are not always certain" (Covington, 1992, p. 160). Marzano (2003) found that role playing and instructional games also serve to motivate, and thus engage, students in the learning process.

Motivating activities include a social aspect that promotes student engagement. Intentional planning for social interaction includes planning for cooperative learning. Cooperative learning consists of "students working together in a group small enough that everyone can participate on a collective task that has been clearly assigned" (Cohen, 1994, p. 3). Barley et al. (2002) found that "[c]ooperative learning, when rigorously implemented, can provide [at-risk] students with enriched instruction through peer interaction resulting in improved student achievement" (p. 60).

Rigorous implementation of cooperative learning must address the clarity of directions that are provided to the student. (Repman, 1993 and Barley et al., 2002). Because struggling students tend to be more passive during group learning situations (Repman, 1993 and Barley et al., 2002), research supports arranging for peer groups to generate solutions rather than having individuals generate solutions during the learning process (Barley et al., 2002). Secada and De La Cruz found that

> [F]or language minority students in particular, the opportunity to discuss mathematics in a small group may precede competent participation in large group discussion. Studies comparing students' communication in their two languages, in large group discussion and in small groups, have found that language minority students display the lowest level of competency when talking in English during large group discussions, frequently leading to underestimation of children's academic competency (as cited in Brenner, 1998).

Research studies stress the importance of the processes related to mathematical learning that occurs in groups while underscoring the teacher's role as critical to student learning in groups (Barley et al., 2002).

The classroom environment influences the affect of the student, which in turn affects how well the student will engage with his peers and with mathematical learning. When students' affective needs are addressed, student engagement is encouraged. For the English language learners, a learning atmosphere and physical environment that fosters student engagement encourages self-expression and provides positive recognition of students' effort and thinking, builds student confidence in mathematical proficiency, and fosters an emotionally safe environment that fosters security for thinking and risk-taking for learning (Huling & Beck, 2005). The classroom environment is visually rich, using non-linguistic and linguistic representations to reinforce math-specific vocabulary and concepts (Huling & Beck, 2005; Marzano, 2001). The physical room arrangement facilitates student interaction and group work (Huling & Beck, 2005). Such interactions in turn influence instructional conversation. Instructional conversation improves mathematical proficiency.

Conclusion

To engage students in the learning process, the teacher should use problems and relevant topics that are inviting to the student. The teacher should use instructional conversations. These efforts allow the teacher to assess a student's entry knowledge given the concept or process to be studied.

Complex, rigorous academic tasks provide opportunities for students to develop conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. While completing these tasks, students should engage in instructional conversation with their peers and with their teacher. These instructional conversations offer opportunities for the students to explore mathematical activity at a deeper level while providing insight for the teacher and the students into the students' level of understanding. As students work on academic tasks, collaborative efforts in small groups provide safe environments for students to learn. Students have the opportunity to explore and communicate about the mathematics being learned in a smaller setting that leads to greater comfort in discussing the mathematics in a whole-group setting.

Planning for engagement and tasks should be purposeful. This planning should reflect a coherent sequence for learning, with well-structured tasks to assist students in developing mathematical understanding and understanding of academic processes. The teacher supports students in these tasks by tending to entry knowledge and building background knowledge by addressing mathematical vocabulary. Teachers model, explain, prompt, and discuss thought processes about mathematical content and critical thinking. As they do this, teachers should provide opportunities for students to refine and reflect upon their conceptual and procedural understandings. As at-risk learners and English language learners refine their understanding, they extend their understandings. They also make connections between their new understanding and their entry knowledge.

Teachers should assess student understanding at formative and summative points during instruction. As the teacher assesses student understanding, he or she should provide feedback to the student about growth. The teacher should also use knowledge gained during the assessment process to monitor the effectiveness of instruction, adjusting as needed to increase student success.

PROJECT DESCRIPTION

Overview

The *Maximizing Algebra II Performance (MAP)* Institute is a research-based professional development opportunity for Algebra II mathematics educators. The primary focus of *MAP* is to educate teachers about the inherent alignment between concepts and their connections to other concepts or big ideas vertically with a grade band as well as other strands within the grade level.

MAP utilizes the 5E instructional model, an inquiry-based model of instruction. The 5E lesson structure offers well-timed opportunities to incorporate instructional strategies, such as cooperative learning, vocabulary development, and questioning techniques, that have been proven to impact student achievement. The 5E instructional model encourages a consistent structure for learning with characteristic activities during each phase, so that students can monitor the learning process and gain metacognitive knowledge. The 5E instructional model supports the structure of learning as described by Marzano (2003). Included in the professional development materials are student-ready lessons for each grade level addressed. All student lessons are written using the 5E instructional model. During these lessons, students are engaged in an intensive, rich mathematical experience at the level of cognitive rigor that is demanded by TAKS while addressing the needs of at-risk learners and English language learners.

Instructional Framework

The framework upon *MAP* is built is the 5E instructional model. This model provides the best structure to support the research-based instructional strategies that positively impact a student's mathematical proficiency.

- **ENGAGE:** The instructor initiates this phase by asking well-chosen questions, by presenting a problem to be solved, or by showing something intriguing. The activity should be designed to interest students in the problem and to make connections between past and present learning. The Engage phase of the instructional model facilitates building common ground for all students.

  Haynes (n.d.) and Jarrett (1999) propose that English Language Learners (ELL) at the Beginning and Intermediate levels of language proficiency still rely heavily on prior knowledge. They also state that ELL students determined to be at the Advanced and Advanced High levels of English acquisition also benefit from connections to prior knowledge and setting the stage for learning. The Engage phase benefits these students by connecting prior knowledge from past learning to the posed questions, problem, or engaging activity.

- **EXPLORE:** The exploration phase provides the opportunity for students to become directly involved with the key concepts of the lesson through guided exploration that requires them to probe, inquire, and question. As we learn, the puzzle pieces (ideas and concepts necessary to solve the problem) begin to fit together or have to be broken down and reconstructed several times. In this phase, instructors observe and listen to students as they interact with each other and the activity. Instructors provide probing questions to help students clarify their understanding of major concepts and redirect the questions when necessary.

Jarrett (1999) and Haynes (n.d.) also agree that providing opportunities to explore using concrete models, visuals, patterning, as well as mathematical representations accelerates learning and increases retention for ELL students. The opportunities found in the Explore phase allow instructors to observe and listen to ELL students to determine misconceptions that are language-based and misconceptions that are mathematics-based.

- **EXPLAIN:** In the explanation phase, collaborative learning teams begin to logically sequence events and facts from the investigation and communicate these findings to each other and the instructor. The instructor, acting in a facilitation role, uses this phase to offer further explanation and provide additional meaning or information, such as formalizing correct terminology. Giving labels or correct terminology is far more meaningful and helpful in retention if it is done after the learner has had a direct experience. The explanation phase is used to record the learner's development and grasp of the key ideas and concepts of the lesson.

  Barton and Heidema (2002) emphasized the importance of moving ELL from basic interpersonal communication skills to cognitive academic language proficiency. Mathematics is a language in and of itself. The Explain phase of the instructional model provides structure for facilitating the transition from interpersonal communication to mathematical academic language.

- **ELABORATE:** The elaboration phase allows for students to extend and expand what they have learned in the first three phases and connect this knowledge with their prior learning to create understanding. It is critical that instructors verify students' understanding during this phase.

  The National Council of Teachers of Mathematics states that all students should have equitable and optimal opportunities to learn challenging mathematics free from racial, gender, socioeconomic status, or language bias. Secada and De La Cruz (1996) found that extending student's knowledge from prior exploration provided optimal instructional opportunities for students to acquire academic proficiency. The Elaborate phase provides these additional learning opportunities.

- **EVALUATE:** Throughout the learning experience, the ongoing process of evaluation allows the instructor to determine whether the learner has reached the desired level of understanding the key ideas and concepts. More formal evaluation can be conducted at this phase (Bybee, 1997).

The 5E instructional model for an inquiry-based lesson fosters strategies that have been shown to impact student achievement. For example, *questioning strategies* are embedded in each phase of the lesson through "Facilitation Questions" provided for the teacher. The teacher poses these questions to students who are struggling with the lesson to guide their thinking. These questions are designed to prompt independent student thinking so that students may engage in instructional conversations about what they already know about the concepts and procedures, about what they are learning, and about their progress toward mathematical proficiency (CREDE, 1997). *Cooperative learning strategies* are also embedded in the lessons whenever enriched instruction through peer interaction is needed.

Marzano, Pickering, and Pollock (2001) in their book, *Classroom Instruction that Works,* identify nine categories of strategies that have shown to have an effect on student achievement. The nine categories are:

- Identifying similarities and differences
- Summarizing and note taking
- Reinforcing effort and providing recognition
- Homework and practice
- Nonlinguistic representations
- Cooperative learning
- Setting objectives and providing feedback
- Generating and testing hypotheses
- Questions, cues, and advance organizers

The authors suggest three phases educators might include in an instructional unit to utilize the nine categories of strategies. Instructors should begin a unit of instruction by utilizing strategies and well designed questions to provide students an opportunity to connect prior experiences with present learning. The instructor continues to directly involve the students with key concepts of the lesson through guided exploration. The first phase of Marzano, Pickering, and Pollock's instructional planning parallel the Engage and Explore phase of the 5E Instructional model. Students connect prior experiences through well designed questions that focus student's attention and directly involved the student in the learning process.

The second phase included in *Classroom Instruction that Works* is considered to occur during the unit. New knowledge is introduced, students are provided an opportunity to apply new knowledge gained in the unit, and instructors monitor students' attainment of learning goals. Cooperative learning, guided discovery, or any strategies included in Marzano, Pickering, and Pollock's nine categories are encouraged in the lesson design. The 5E instructional model provides a well defined structure for educators to ensure research-based strategies are deployed in a well designed and fluid process to maximize student achievement. Marzano, Pickering, and Pollock's second phase parallel the Explore, Explain, and Elaborate phase of the 5E instructional model.

The third phase of Marzano, Pickering, and Pollock's planning model is defined as the end of the unit which parallels the Evaluate phase of the 5E model. Although instructors are constantly monitoring the progress of their students, the end of the unit serves as the formal evaluation of student learning.

Any one strategy will not effectively work with all students all the time. The model facilitates strategic planning by allowing educators to redefine a lesson as a learning cycle that include a variety of researched-based instructional strategies. The 5E Instructional model for lesson design provides the structure that naturally incorporates researched-based strategies. Good teaching is good teaching for all students.

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Implementation Strategies

MAP: Considering the Needs of Different Districts and Audiences

Working with Limited Days

Number of Days	Possible Sequence
1	• Engage (45 minutes) • Explore/Explain/ Elaborate Some combination of two depending on scope and sequence needs of audience (4.5 hours). • Evaluate (45 minutes)
2	As outlined in presenter materials
3	Day 1: • Engage (45 minutes) • Explore/Explain/Elaborate 1 (2.5 hours) • Explore/Explain/Elaborate 2 (2.5 hours) • Homework: Implement lesson. Bring back student work. (15 minutes) Day 2: • Analyze student work (45 minutes) • Explore/Explain/Elaborate 3 (2.5 hours) • Explore/Explain/Elaborate 4 (2.5 hours) • Homework: Implement lesson. Bring back student work. (15 minutes) Day 3: • Analyze student work (1 hour) • Evaluate – (1 hour) • Apply processing model and 5E instructional model to upcoming unit of instruction (4 hours)

Working with Limited Hours

Number of Hours	Possible Sequence
1	Student lesson appropriate to audience
2	Student lesson appropriate to audience
3	Two student lessons appropriate to audience
4	Explore/Explain/Elaborate 1, 2, 3, or 4

Master Materials List

(for a group of approximately 40 participants)

Sticky notes – 20 pads
Rulers – 40
Highlighters – 40
Tape – 10 rolls
Flip chart markers – 10 sets
Transparency markers
Transparencies – 1 box
Scissors – 10 pair
Tape measures (metric) – 10
Masking tape – 1 roll
Pencils – 40
Chart paper
Stopwatches – 10
Graphing calculators – 40
Meter sticks – 10
String – 1 ball
CBR – 10
Linking Cables – 10
Water bottles (500ml) – 10
Rubber bands
Linguine (1 package)
Film canisters (empty) – 10
Pennies – approximately 50
Patty paper
Teach Timer™

Engage

Leader Notes: Lesson Learned?

Purpose:

The purpose of the Engage phase of the professional development is to connect participants' prior knowledge of teaching and learning related to Algebra 2 TEKS to the new learning which will take place in the professional development. To begin discussions related to the big ideas forming the core of this professional development, participants will explore the notion of parent functions from two perspectives: teaching and learning.

The Engage phase also serves as an initial window through which the facilitator can view participants' current understandings and beliefs related to:

- Participants' knowledge of teaching and learning related to parent functions and
- Participants' perceived stumbling blocks regarding the teaching and learning related to parent functions.

Descriptor:

Participants will analyze sets of data based on notions of parent functions. After working with the data sets, participants will work in small groups to study a sample lesson plan that uses the data sets. After analyzing the 5E lesson plan, participants will generate questions that help students focus on attributes of data that may indicate the use of a particular function in the process of generating a model for a set of data. To conclude, participants will generate a list of potential stumbling blocks related to the teaching and learning of parent functions.

Duration:

45 minutes

TEKS:

a5 Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.1 **Foundations for functions.** The student uses properties and attributes of functions and applies functions to problem situations.

2A.1A The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

2A.1B The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

TAKS Objectives Supported:

While the Algebra 2 TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Per Participant: **Setting the Stage**, **Ms. G's Lesson Plan**, **A Teaching Perspective**, and **A Learning Perspective** handouts

Leader Notes:
The purpose of this activity is to engage participants in the study of data with the intent of modeling the data using appropriate functional relationships. Participants will study Mrs. G's 5E lesson. This is a nonjudgmental task. During this phase of the professional development, avoid correcting participants or guiding them to a particular answer. Presenter questions should center on why a participant or group made the choice(s) they did. These questions allow the presenter to understand the set of knowledge and beliefs that participants bring to the training.

Engage

Part 1: Data Analysis (15 minutes)

*Distribute the **Setting the Stage** handout and **Ms. G's Lesson Plan** handout to each participant. Assign each group of participants one set of data to use as a reference as they work through the Case Study Lesson Page.*

Prompt the participants to work through the lesson using the assigned data set. This will allow participants to acquaint themselves with the lesson while the leader is able to gauge participants' knowledge about and comfort with analyzing sets of data. If the individuals in a group are not interacting with each other or seem to be struggling with responding to the questions, use the facilitation questions to help begin, focus, or continue the conversation.

Facilitation Questions

- Why is it important to consider the domain and range of the data?
  Answers may vary.
- Why is it important to consider whether the data are continuous or discrete?
  Answers may vary.
- Is there an interval where the rate of change indicates a particular parent function?
  Answers may vary.

<u>Part 2</u>: A Teaching Perspective (10 minutes)

*Distribute **A Teaching Perspective** handout to each participant. Participants should remain in their initial groups as they respond to the questions posed on the handout. The facilitator should be listening to the participants' responses as a means of gaining insight into beliefs about teaching Algebra II content. If the individuals in a group are not interacting with each other or seem to be struggling with responding to the questions, use the facilitation questions to help begin, focus, or continue the conversation.*

Facilitation Questions

- What questions might we pose to prompt students to explain whether or not the data represent a function?
  Answers may vary.
- What questions might we pose to ensure that the students are interpreting the data in graphical form?
  Answers may vary.
- What questions might we pose to prompt students to consider the importance of intervals when analyzing data?
  Answers may vary.
- What questions might we pose to prompt students to consider rates of change within the data and/or intervals of data?
  Answers may vary.

<u>Part 3</u>: A Learning Perspective (10 minutes)

Distribute ***A Learning Perspective*** *handout to each participant. Participants should remain in their groups as they respond to the questions posed on the handout. The facilitator should redirect any groups that begin to speak negatively about students and their capabilities. The facilitation questions might need to begin with the stem "What would you hope to hear a student…" If the individuals in a group are not interacting with each other or seem to be struggling with responding to the questions, use the facilitation questions to help begin, focus, or continue the conversation.*

Facilitation Questions

- What might a student write or say that would provide insight into his or her understanding about whether or not the data represent a function?
  Answers may vary.
- What might a student write or say that would provide insight into his or her understanding about interpreting the data in graphical form?
  Answers may vary.
- What might a student write or say that would provide insight into his or her understanding about the importance of intervals when analyzing data?
  Answers may vary.
- What might a student write or say that would provide insight into his or her understanding about rates of change within the data and/or intervals of data?
  Answers may vary.

<u>Part 4</u>: Reflection (10 minutes)
In a whole group setting, prompt participants to share their responses to the following questions. Simultaneously scribe the group's responses on two sheets of chart paper (Stumbling Blocks to Teaching and Stumbling Blocks to Learning) for use in the Evaluate Phase of the professional development. Answers may vary widely depending on the make-up of your group of participants.

1. **What opportunities for student interaction does Mrs. G provide?**
   As participants respond ask if their response should be listed as a stumbling block to teaching or a stumbling block to learning.

2. **How is Mrs. G supporting the needs of her English Language Learners, Students with Special Needs, and At-Risk Students?**
   As participants respond ask if their response should be listed as a stumbling block to teaching or a stumbling block to learning.

3. **How do we balance the need to review and refresh with the demands of new instruction in Algebra II? What are the implications of that balance?**
   As participants respond ask if their response should be listed as a stumbling block to teaching or a stumbling block to learning.

4. **Are there any additional stumbling blocks you would like to add?**
   As participants respond ask if their response should be listed as a stumbling block to teaching or a stumbling block to learning.

5. **Are these stumbling blocks only applicable to parent functions? Why?**
   Answers may vary.

6. **Why begin a professional development with a discussion about stumbling blocks?**
   Answers may vary. Beginning with a short discussion of stumbling blocks allows the facilitator to "table" what often leads to off-task discussions until the end of the professional development. Many of the strategies used in this professional development will address the stumbling blocks. These stumbling blocks will be part of the reflective process in the Evaluate phase.

While the answers to the questions posed in the Engage phase will vary, responses open a window into participants' cognitive, pedagogical, and philosophical beliefs. The facilitator listens carefully in a nonjudgmental fashion and should focus on the substance of participant responses related to cognitive, pedagogical, and philosophical beliefs.

Setting the Stage

Ms. G has 8 years of experience teaching Algebra II. This year she has five sections of Algebra II filled with 11th and 12th grade students. Ten percent of her students are enrolled in Algebra II for the second time. Ten percent of her students speak little or no English and have Spanish or Vietnamese as their primary language. Twenty-five percent of her students are fluent in English and Spanish. Five percent of her students are fluent in English and Vietnamese. Ten percent of her students have IEPs requiring accommodations and/or modifications. Thirty percent of her students did not pass the TAKS as tenth graders. Forty percent of her students are classified as low SES. Ms. G is sometimes overwhelmed with finding approaches that address the needs of all of the students in her classes.

Ms. G prefers to plan whole group instruction. She is trying to incorporate small group instruction a few times each six weeks. Ms. G feels that she gets the best feedback from students when everyone is focused on one task together. She has recently started to have her students write about the mathematics they are studying.

Prior to this lesson, Ms. G reviewed linear, quadratic, and exponential parent functions to help students remember what they learned in Algebra I. She provided a brief explanation of the absolute value parent function as a non-example of the other functions. The students used graphing technology to graph the parent functions, perform transformations on the parent functions, and analyze the rates of change associated with the parent functions.

Ms. G's Lesson Plan

TEKS:

2A.1A The student uses properties and attributes of functions and applies functions to problem situations. The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

2A.1B The student uses properties and attributes of functions and applies functions to problem situations. The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

Engage:
The U.S. Geological Survey (USGS) is dedicated to the timely, relevant, and impartial study of the landscape, our natural resources, and the natural hazards that threaten us. To accomplish this, the USGS collects real-time data about the depth of water in streams, bayous, ponds, and lakes in the United States. Why might they do this?
http://www.usgs.gov/

Explore:
1. Assign each group one of the four sets of data. Provide each group with the appropriate graph and table of data.
2. Prompt students to answer the questions on the activity page for their assigned graph and table of data.
3. Prompt students to create a poster that summarizes their learning.

Explain:
1. Have students share their group summaries.
2. Debrief using these questions.
   a. How did the reasonable domains compare for these different situations?
   b. How did the reasonable ranges compare for these different situations?
   c. Are the data continuous or discrete? Why?
   d. Which, if any, parts of these graphs can be modeled by a parent function? Why?

Elaborate:
1. Assign each group a different set of data.
2. Direct each group to analyze this new set of data. How does it compare to your original set of data? How is it different?

Evaluate:
Prompt students to complete 2 of the 3 sentence starters.
1. Understanding domain and range helps me to…
2. A parent function tells…
3. The rate of change of data helps us identify a possible parent function because…

Ms. G's Lesson Plan: Student Activity Page

1. What is the title of your set of data?

2. What do you notice about the graph of the data?

3. What is a reasonable domain for this situation? Why?

4. What is a reasonable range for this situation? Why?

5. Are the data continuous or discrete? Why?

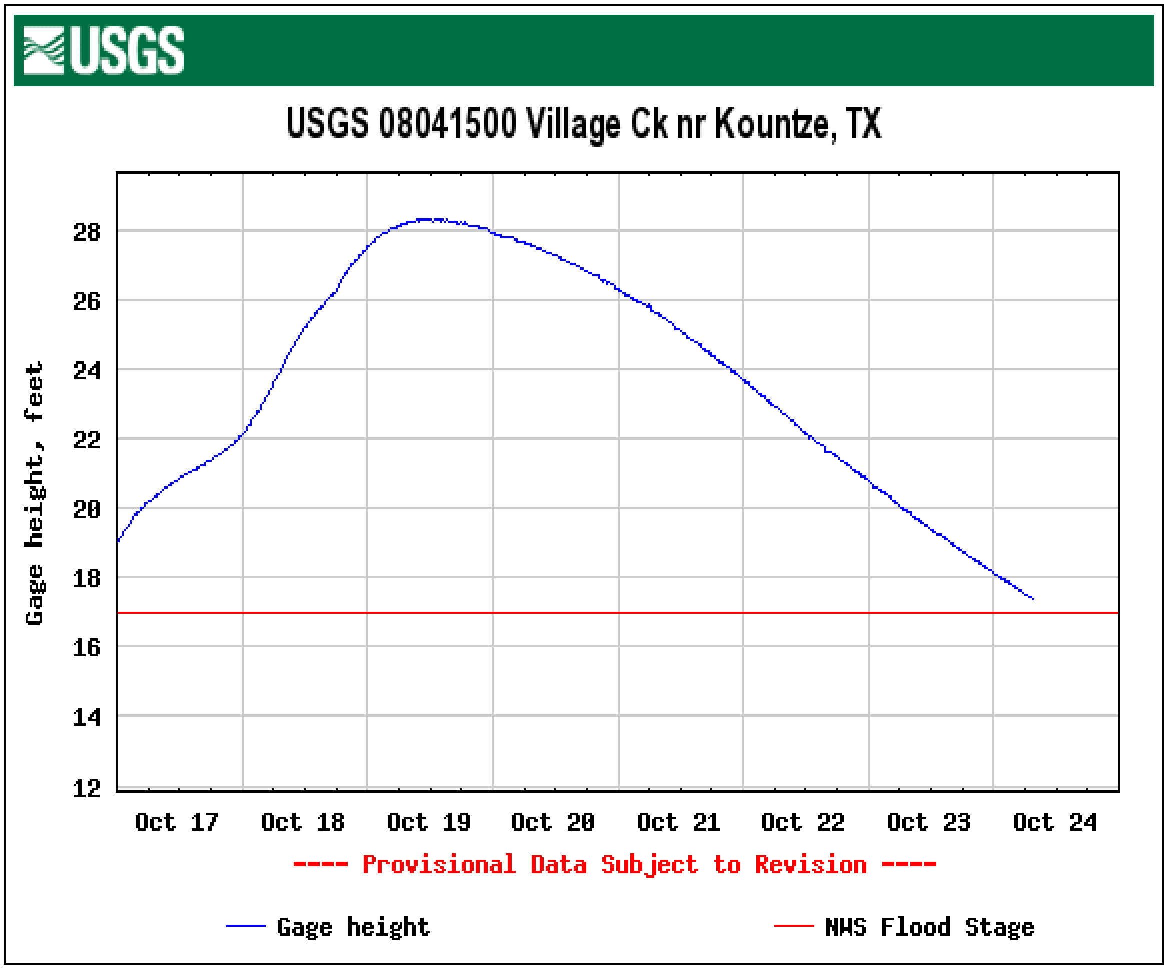
6. Which, if any, part of this graph can be modeled by a linear function? How do you know?

7. Which, if any, part of this graph can be modeled by a quadratic function? How do you know?

8. Which, if any, part of this graph can be modeled by an exponential function? How do you know?

9. Which, if any, part of this graph can be modeled by an absolute value function? How do you know?

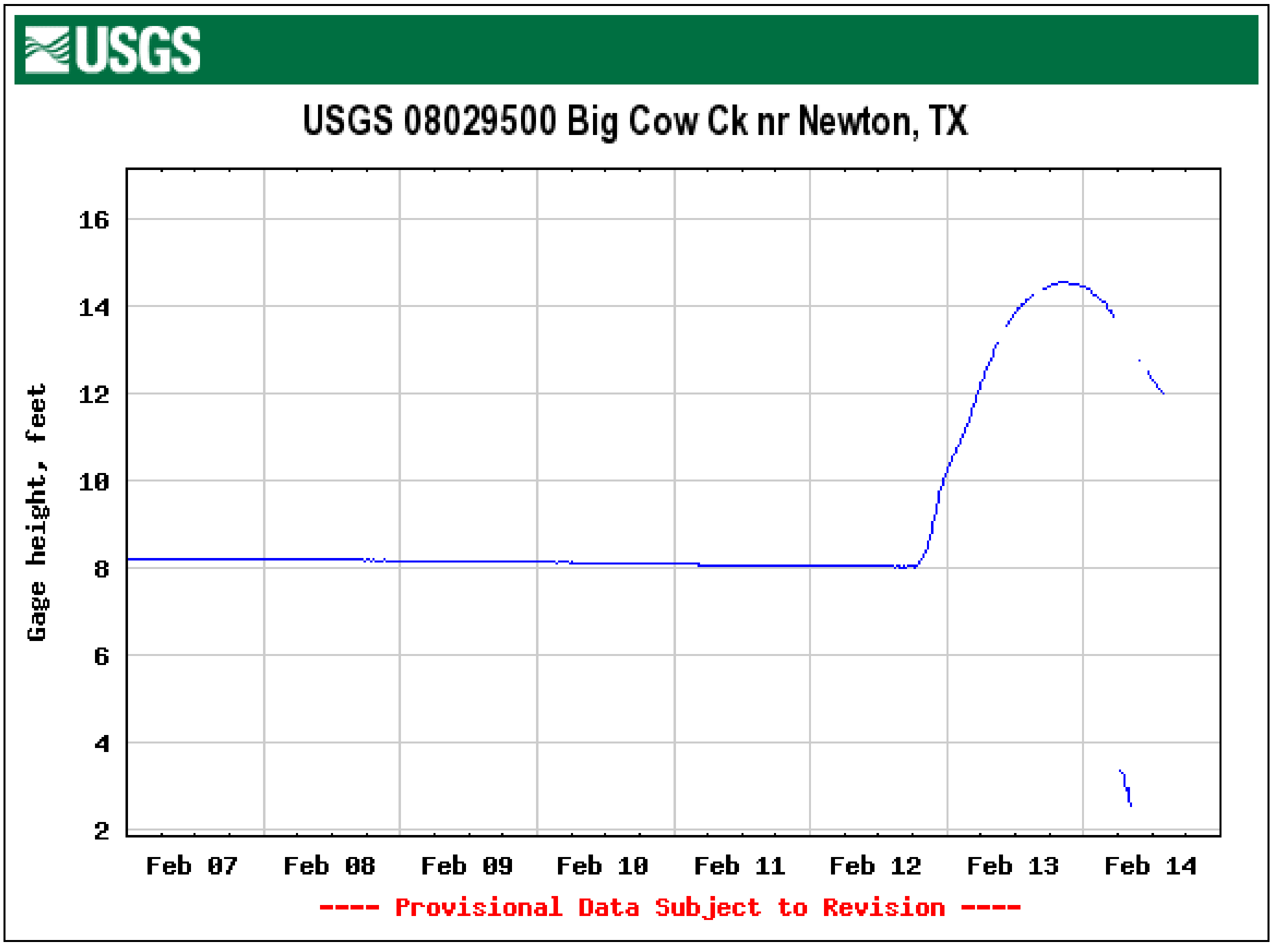
Ms. G's Lesson Plan: Data Set A



Date	Gage Height at Noon (ft)
Oct. 17	20.8
Oct. 18	25.4
Oct. 19	28.2
Oct. 20	27.1
Oct. 21	25.0
Oct. 22	22.1
Oct. 23	19.2

http://waterdata.usgs.gov/nwis/rt

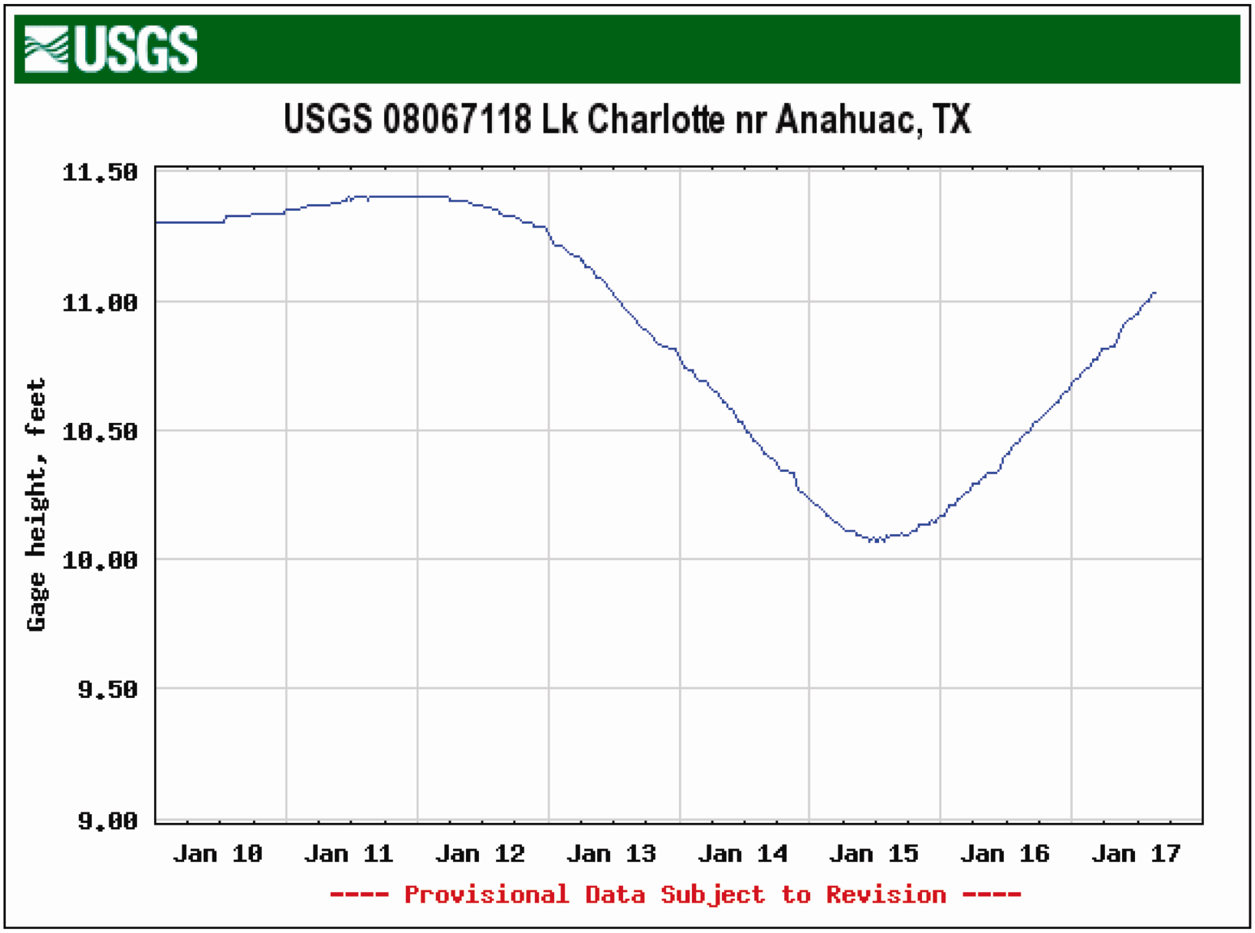
Ms. G's Lesson Plan: Data Set B



Date	Gage Height at Noon (ft)
Feb. 7	8.2
Feb. 8	8.2
Feb. 9	8.2
Feb. 10	8.1
Feb. 11	8.0
Feb. 12	8.0
Feb. 13	13.5
Feb. 14	12.1

http://waterdata.usgs.gov/nwis/rt

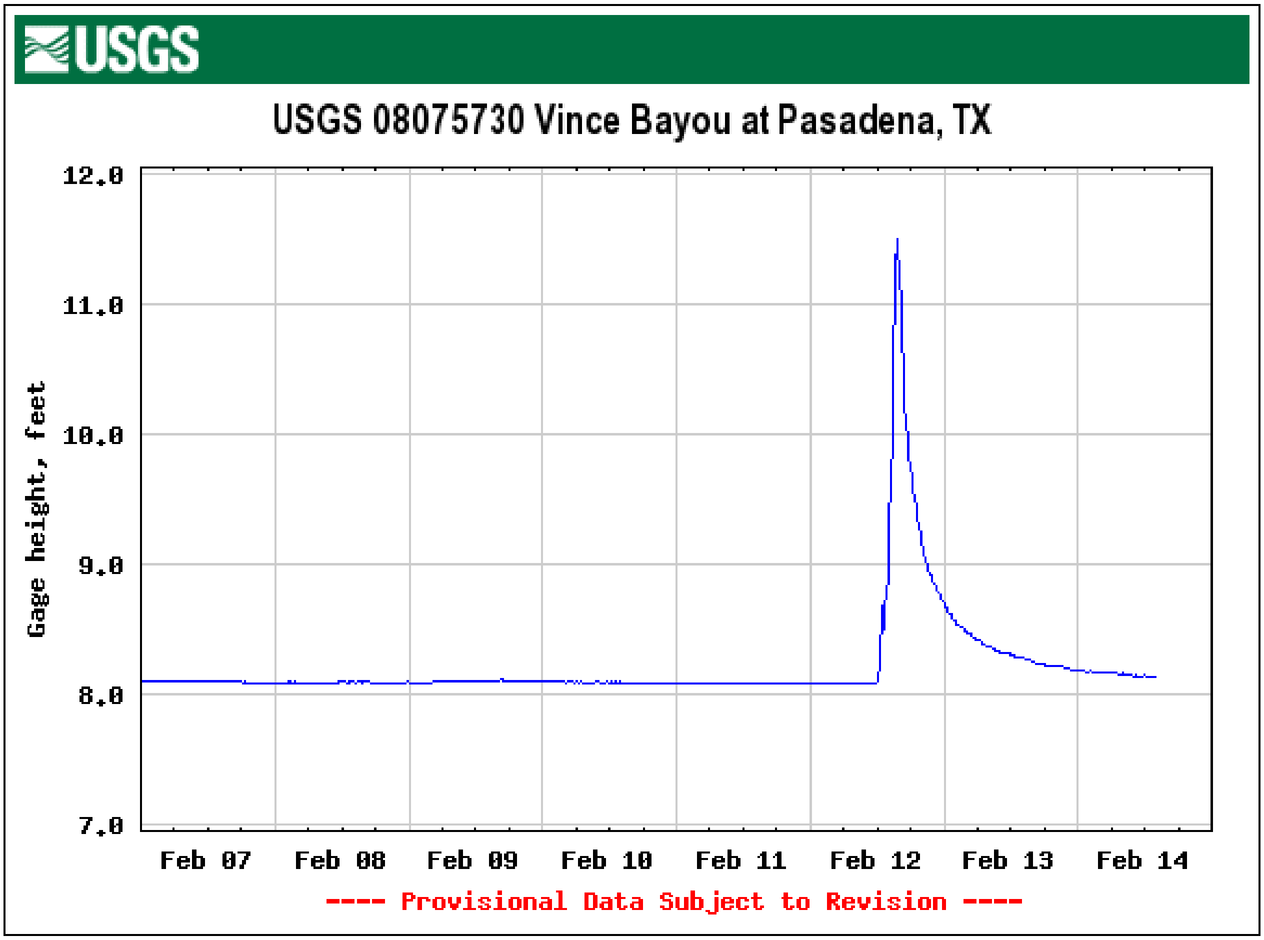
Ms. G's Lesson Plan: Data Set C



Date	Gage Height at Noon (ft)
Jan. 10	11.30
Jan. 11	11.35
Jan. 12	11.32
Jan. 13	11.08
Jan. 14	10.51
Jan. 15	10.10
Jan. 16	10.40
Jan. 17	10.90

http://waterdata.usgs.gov/nwis/rt

Ms. G's Lesson Plan: Data Set D



Date	Gage Height at Noon (ft)
Feb. 7	8.1
Feb. 8	8.1
Feb. 9	8.1
Feb. 10	8.1
Feb. 11	8.1
Feb. 12	8.1
Feb. 13	8.3
Feb. 14	8.2

http://waterdata.usgs.gov/nwis/rt

A Teaching Perspective

1. How does this lesson teach students to become better problem solvers?

2. How does this lesson teach students to communicate mathematics?

3. How does this lesson teach students to reason mathematically?

4. What questions should the teacher pose during this lesson?

5. What teaching suggestions would you offer this teacher?

A Learning Perspective

1. What should be the learning outcomes for this lesson?

2. What prior learning is needed for students to be successful with this lesson?

3. What evidence of learning is provided by this lesson?

4. What evidence of learning is lacking?

5. What misconceptions might students develop based on this lesson?

6. What about this lesson helps students learn to value mathematics?

7. How does this lesson help students become more confident in their ability to do mathematics?

Leader Notes: Backwards and Forwards

Purpose:

In this Explore phase participants are investigating multiple ways (concretely, tabularly, graphically, symbolically, etc.) of teaching the concepts and procedures of inverse functions.

Descriptor:

Participants will explore inverses of functions concretely using patty paper to perform reflections of the graph of a relation, numerically by examining number patterns found in tables of data values, then symbolically determining the inverse of a given function and whether or not two relations or functions are inverses of each other.

Duration:

3 hours

TEKS:

a5 Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.1 **Foundations for functions.** The student uses properties and attributes of functions and applies functions to problem situations.

2A.1A The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations;

2A.1B The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

2A.2 **Foundations for functions.** The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

2A.2A The student is expected to use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

2A.4 **Algebra and geometry.** The student connects algebraic and geometric representations of functions.

2A.4A The student is expected to identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

2A.4B The student is expected to extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

2A.4C The student is expected to describe and analyze the relationship between a function and its inverse.

TAKS™ Objectives Supported:

While the Algebra 2 TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 5: Quadratic Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance: Poster-size version of **Processing Framework Model**, **Vocabulary Model** template copied onto cardstock or colored paper

Presenter Materials: blank transparency (optional – chart paper could be used) and transparency pen(s),

Per group: chart paper, markers

Per participant: graphing calculator, several sheets of patty paper, sticky notes, copies of participants' pages

Leader Notes:

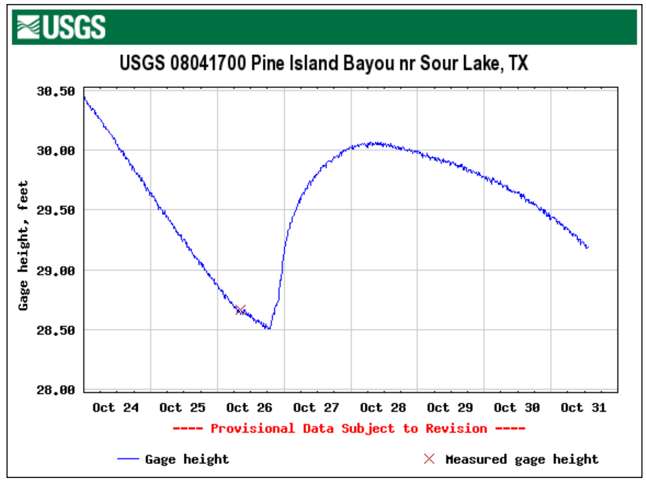
Many participants will likely be familiar with procedures associated with inverses of functions. During this portion of the professional development, be sure to emphasize conceptual understanding of inverses of functions. Connect the known procedures with each other and anchor them in conceptual underpinnings.

In a typical workshop setting, Parts 1, 2, and 3 can be done consecutively with discussion at the end of each part. Parts 4 and 5 can be jigsawed among groups; after the jigsaw is completed, then collectively draw generalizations about inverses of functions. Parts 6 and 7 (which deal with compositions) can serve as extensions of inverses into Precalculus notions of composition of functions.

Explore

<u>Part 1</u>: Generating an Inverse Relation

Note to Leader: At the end of Part 1, participants should be able to recognize inverse relations graphically as a reflection across the line $y = x$.



Post the scatterplot of gage height vs. time for Pine Island Bayou for participants to view. You may wish to use Transparency: Pine Island Bayou. Or, you may wish to copy the image into a PowerPoint presentation.

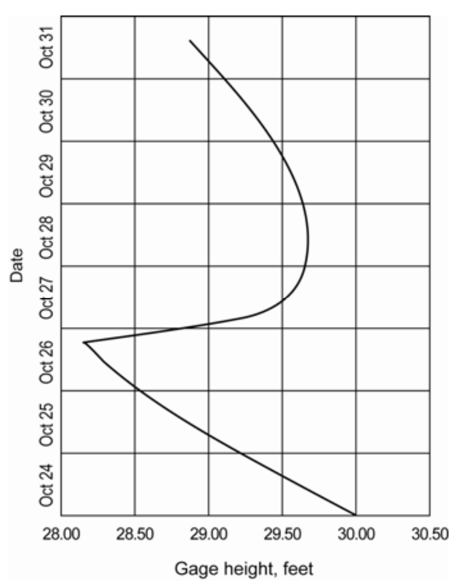
During the Engage phase, you examined graphs of streams' gage heights versus time. Sometimes, hydrologists are concerned about a particular gage height and when the gage for the stream measured that height. In those cases, it makes sense to consider a plot of time versus gage height.

Note to Leader: Recall that scatterplots of data are described as plots of the dependent variable versus the independent variable. In this example, gage height depends on time, so the scatterplot is described as gage height versus time.

1. **For the scatterplot of gage height versus time, what are the inputs and outputs?**
   Time is the input and gage height is the output.

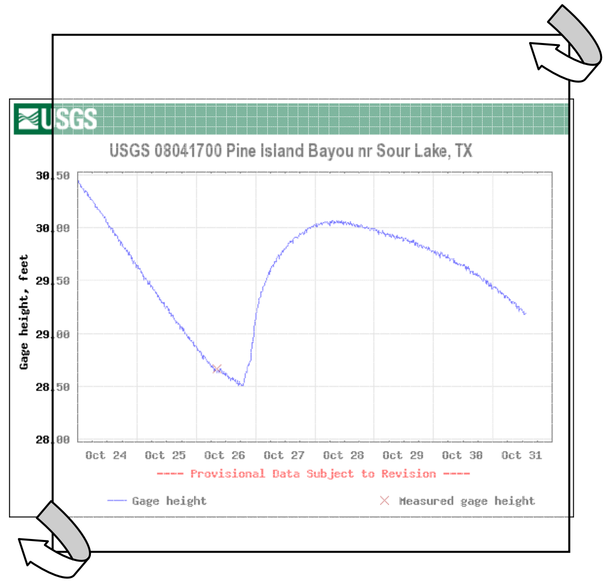
2. **Does this scatterplot represent a functional relationship? How can you tell?**
   Yes. For each x-value (time) there is only one corresponding y-value (gage height).

3. **What would the graph of the same data look like if we plotted time versus gage height? Sketch a prediction of the graph.**
   Responses may vary. Sample response:



Trace the graph of gage height versus time onto a sheet of patty paper. Be sure to trace the plot and the axes then label your axes on your patty paper.

Take the top-right corner in your right hand and the bottom-left corner in your left hand and flip the patty paper over. Align the origin on the patty paper with the origin of your original graph and align the axes on the patty paper with the axes on the graph.

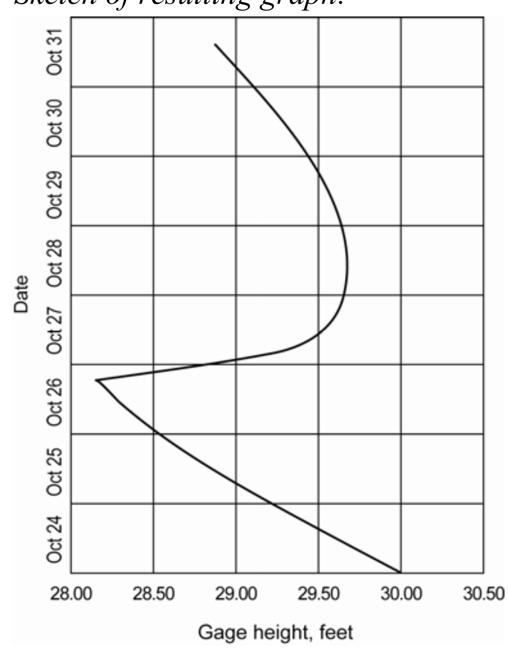


Note to Leader: One purpose of this part of the Explore phase is to describe in a non-numerical way the effect of generating an inverse relation from a given set of data. Participants may express concern that (1) the grid is not square; i.e., the gridlines from the *x*- and *y*-axes are not equidistant, and (2) there is no apparent origin of the graph. Assure them that for our purposes in this activity, it is OK.

Also, at this point some participants may question the absence of the vocabulary word "reflection" for this particular action. In this part of the Explore, we are developing the notion that the inverse is a reflection of a graph across the line $y = x$. The term "reflection" as it is used in this context will be formalized later.

4. **Sketch the resulting graph.**
   Responses may vary. Sample response:
   Patty paper reflection:

Sketch of resulting graph:



5. **How is the new graph similar to the original graph? How are they different?**
Responses may vary. Sample responses may include: The x- and y-axes changed places. The x-axis became the vertical axis. The y-axis became the horizontal axis. The x-axis became the gage height. The y-axis became the date. One graph represents a function, the other does not. The domain became the range, the range became the domain.

6. **Is the new graph a function? How do you know? Given the situation, does it make sense for the graph to be or not to be a function? Explain your reasoning.**
The new graph is not a function because there are multiple y-values for the same x-value (it fails the vertical line test). It makes sense that the new graph is not a function because there are various dates when the stream height is 29 feet, for example.

On the activity sheet, "Know When to Fold 'Em," trace each graph (both curves) onto a piece of patty paper. Be sure to also trace and label the axes.

Fold the patty paper to find all possible lines of symmetry for each pair of graphs shown. Identify the equation for each possible line of symmetry. Record the number of lines of symmetry and their equations for each graph on your recording sheet.

Possible responses:

	Total Number of Lines of Symmetry	**Equation(s) of line(s) of symmetry**
Graph 1	*2*	$y = x$ *and* $y = -x - 5$
Graph 2	*1*	$y = x$
Graph 3	*1*	$y = x$
Graph 4	*1*	$y = x$
Graph 5	*1*	$y = x$
Graph 6	*1*	$y = x$

7. **What patterns do you notice among the lines of symmetry for each of the graphs?**
Responses may vary. Participants should notice that for each of the given pairs of graphs, the line $y = x$ *was a line of symmetry.*

8. **Which transformation describes the folds across a line of symmetry for your graphs?**
Each fold across a line of symmetry reveals a reflection across the line of symmetry.

9. **Make a summary statement describing the relationship between each pair of curves in the set of given graphs.**
Responses may vary. Participants should observe that the pair of curves on each graph is a reflection across the line $y = x$.

Facilitation Question

- How are the curves related in terms of their lines of symmetry?
  Each curve is a reflection of the other across the line of symmetry ($y = x$*).*

Note to Leader: Discuss participants' summary statements as a whole group before proceeding to Part 2.

Facilitation Questions

- How are the curves related graphically?
  Each curve is a reflection of the other across the line of symmetry ($y = x$).
- What happens to the x-axis and y-axis when the graphs are reflected across their line(s) of symmetry?
  The x-axis becomes a vertical axis and the y-axis becomes a horizontal axis.

<u>Part 2</u>: Numerical and Symbolic Representations

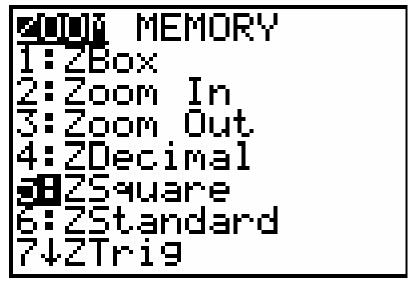
Note to Leader: At the end of Part 2, participants should understand that inverse relations have domains and ranges that are reversed (i.e., the domain of one relation is the range of the other and the range of one relation is the domain of the other). Participants should connect tabular (numerical), mapping, graphical, and symbolic representations of inverse relations.

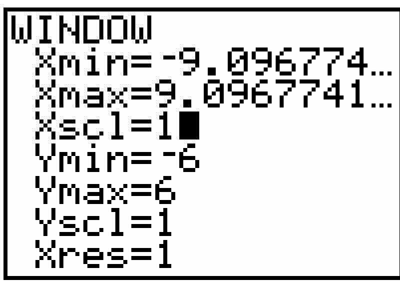
The following coordinates were used to create a geometric design for a quilting pattern.

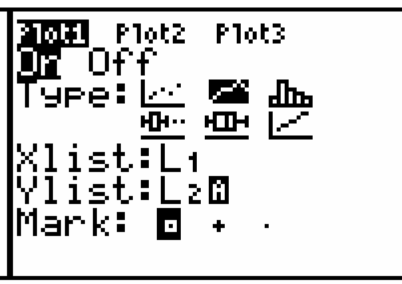
List 1	1	1	0	0	−1	−1	−2	−1	−2	−1	1
List 2	2	4	5	4	5	4	4	3	3	2	2

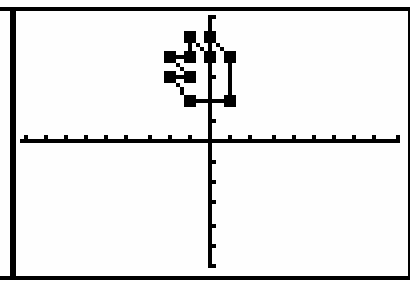
1. **Create a connected scatterplot of L_2 versus L_1. Sketch your graph and describe your viewing window.**

   TIP: Be sure to use a square window. After selecting your appropriate domain and range, use the Zoom-Square feature to square out the grid in your viewing window.









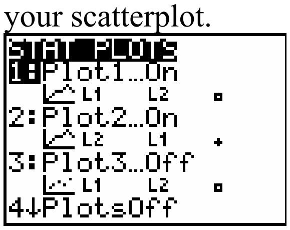
Facilitation Question

- What are the independent and dependent variables in this situation?
  Though there is no clear dependency relationship, we are treating List 1 as independent and List 2 as dependent.

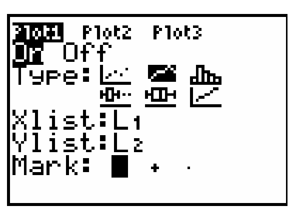
Technology Facilitation Tip:

A connected scatterplot can be created from the STAT PLOT menu:

Press 2nd Y= to select `STAT PLOT`. Use the ▲▼ arrow keys to select your scatterplot.

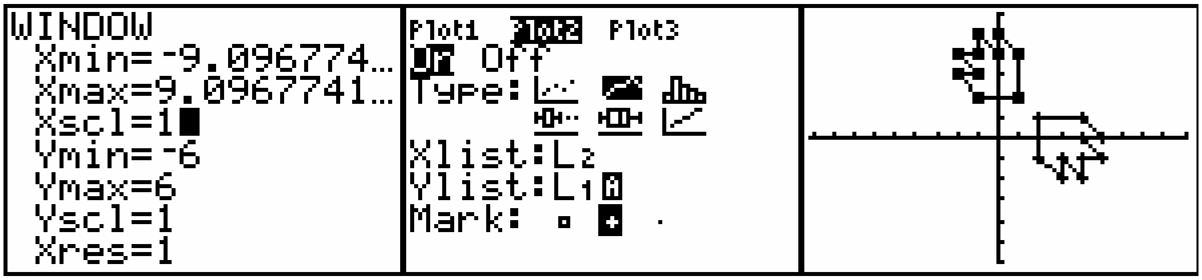


Use the ▲▼ arrow keys to change the `Type` to connected-dot.

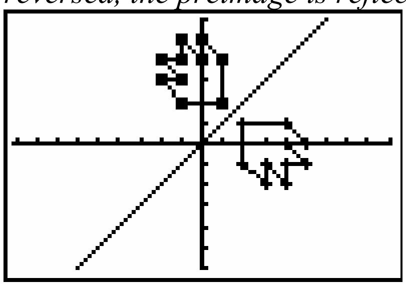


2. **What do you think would happen to the graph if we reversed the *x*- and *y*-values?**
   The graph would reflect across the line $y = x$.

3. Create a second connected scatterplot of L_1 versus L_2. Use a different plot symbol for this scatterplot. Graph this scatterplot with your original scatterplot. Sketch your graph and describe your viewing window. You may need to re-square your viewing window.



4. Compare the two scatterplots. How are they alike? How are they different?
Responses may vary. Participants should observe that when the x- and y-coordinates are reversed, the preimage is reflected across the line y = x.



Facilitation Questions

- Are the two images congruent or similar? How can you tell?
  The images appear to be both congruent and similar (recall that similarity is a special case of congruence with a scale factor of 1).

- What type of transformation might generate the second image from the first?
  A reflection across the line y= x.

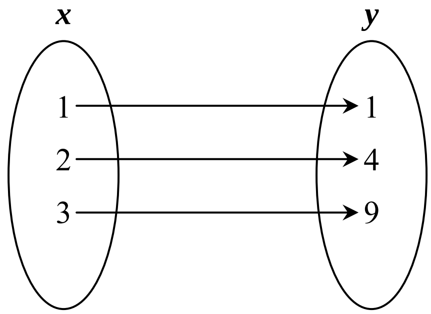
5. How are the *x*- and *y*-coordinates related from the first scatterplot to the second scatterplot? How could you represent this relationship symbolically?
The x-coordinates from the first scatterplot become the y-coordinates for the second scatterplot and the y-coordinates from the first scatterplot become the x-coordinates for the second scatterplot.

Symbolically, this relationship can be represented using the transformation mapping
$T:(x, y) \to (y, x)$

Facilitation Question

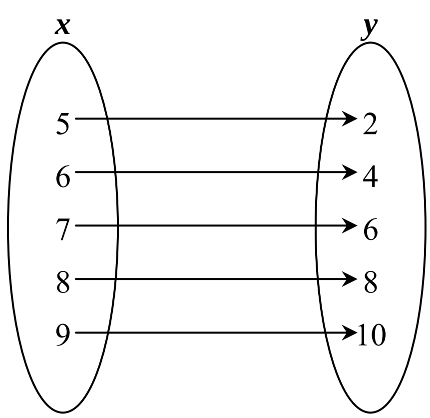
- How can we represent the reversal of *x* and *y* with a transformation mapping?
  $x \to y$ and $y \to x$

Recall that a mapping shows how domain elements for a relation relate, or "map to," their corresponding range elements. For example, the following mapping shows how the *x*-values {1, 2, 3} map to their corresponding *y*-values {1, 4, 9} for the function $y = x^2$.

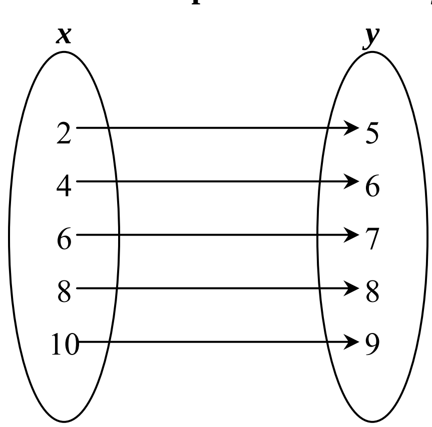


Enter the functions Y1 = 2x – 8 and $\mathbf{Y2 = \frac{1}{2}x + 4}$ into your graphing calculator.

6. Use the table feature of your graphing calculator to generate values for a mapping for Y1 to show the replacement set for *y* when *x* = {5, 6, 7, 8, 9}.
Sample Response:



7. Generate a mapping for Y2 to show the replacement set for *y* when *x* = {2, 4, 6, 8, 10}.



8. How are the two mappings related?
The domain and the range for the two mappings are reversed.

9. In each mapping, to how many *y*-values does any given *x*-value map? Would you expect this to be true for other domain and range elements? How do you know?
Each x-value maps to only one y-value. This would be true for other domain and range elements as well.

10. What does this reveal about the relationships in each mapping?
That each x-value maps to only one y-value confirms that the relationships are functional.

Facilitation Question

- What does it mean about a mathematical relationship when every *x*-value corresponds with only one *y*-value?
Such a correspondence tells us that the relationship is a function.

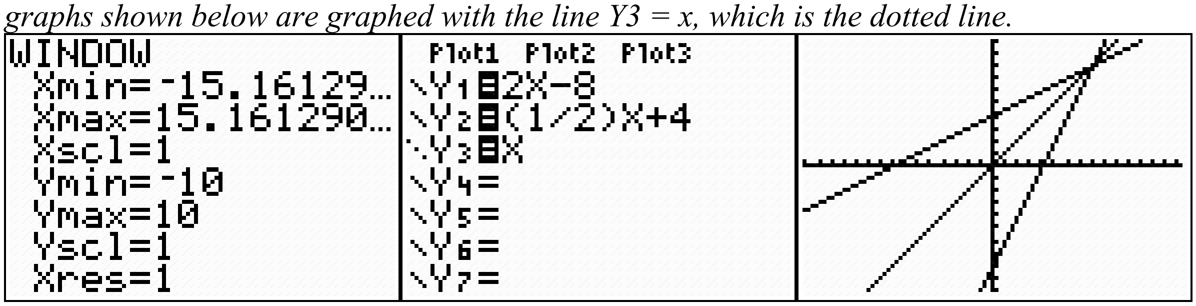
11. In each mapping, how many *x*-values map to any given *y*-value? Would you expect this to be true for other domain and range elements? How do you know?
Only one x-value maps to any given y-value. This would be true for other domain and range elements as well.

12. What does this reveal about the relationships in each mapping?
When only one x-value maps to any given y-value, the relationship is said to be one-to-one.

Facilitation Question

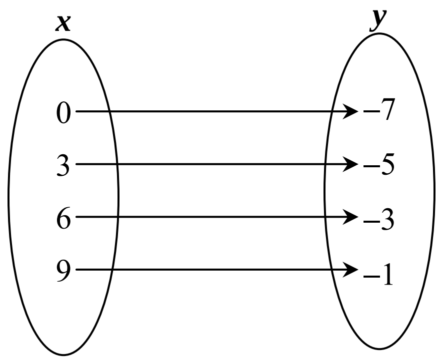
- What does it mean about a mathematical relationship when every *y*-value corresponds to only one *x*-value?
Such a correspondence indicates a one-to-one correspondence, meaning that if you reversed the domain and range, the resulting relation would still be a function.

13. Examine the graphs of Y1 and Y2. How are they related? (Hint: Be sure you are using a square viewing window.)
The graphs of Y1 and Y2 appear to be reflections of one another across the line y = x. The graphs shown below are graphed with the line Y3 = x, which is the dotted line.

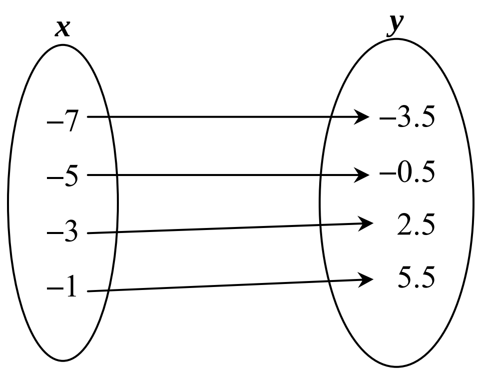


Enter the functions $Y_3 = \frac{2}{3}x - 7$ and $Y_4 = \frac{3}{2}x + 7$ into your graphing calculator.

14. Use the table feature of your graphing calculator to generate a mapping for Y3 to show the replacement set for *y* when *x* = {0, 3, 6, 9}.



15. Use the table feature of your graphing calculator to generate a mapping for Y4 to show the replacement set for *y* when *x* = {–7, –5, –3, –1}.

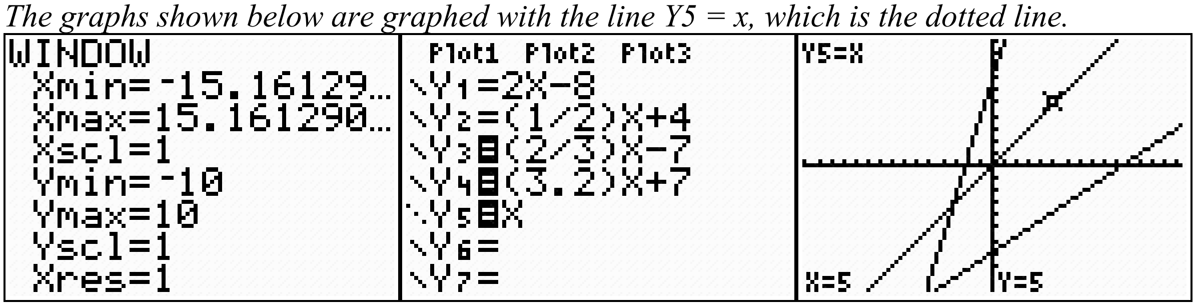


16. How are the two mappings related?

The domain and range are not reversed in these two mappings.

17. Examine the graphs of Y3 and Y4. How are they related? (Hint: Be sure you are using a square viewing window.)

The graphs of Y3 and Y4 do not appear to be reflections of one another across the line y = x. The graphs shown below are graphed with the line Y5 = x, which is the dotted line.



18. The functions in Y1 and Y2 are called "inverse relations" whereas the functions in Y3 and Y4 are not. Based on your mappings, graphs, and equations, why might this be the case?
Inverse relations have domains and ranges that are reversed, as revealed in the mappings. The graphs of inverse relations are reflections across the line $y = x$.

19. Based on your response to the previous question, how might we describe inverse relations graphically, numerically, and symbolically?
Graphically, inverse relations are reflections of each other across the line $y = x$.
Numerically, the domain and range of one relation become the range and domain of the second relation.
Symbolically, x (which represents the domain values) and y (which represents the range values) are interchanged.

Facilitation Questions

- How do the graphical and numerical representations of inverse relations connect to each other?
  Numerically, the x-values and y-values in inverse relations are reversed. Graphically, this reversal results in a reflection of each relation across the line $y = x$.

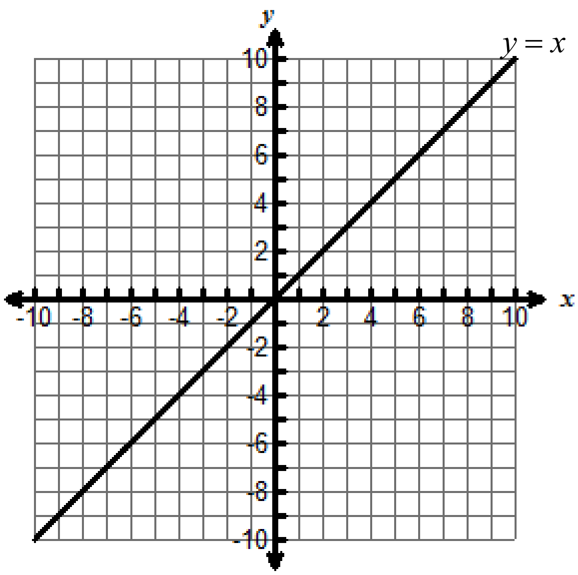
- How do the numerical and symbolic representations of inverse relations connect to each other?
  The symbols x and y are used to represent all domain elements and range elements, respectively. Using a symbol to represent all domain or range elements generalizes the reversal of all domain and range elements to generate an inverse relation.

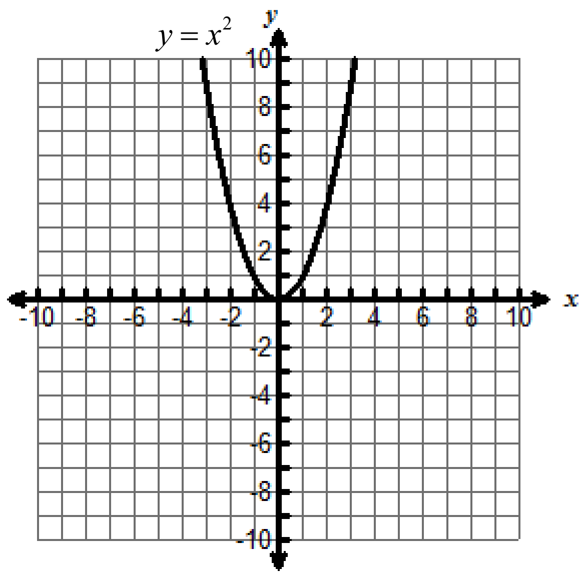
Note to Leader: Discuss participants' responses to the last question before proceeding to Part 3.

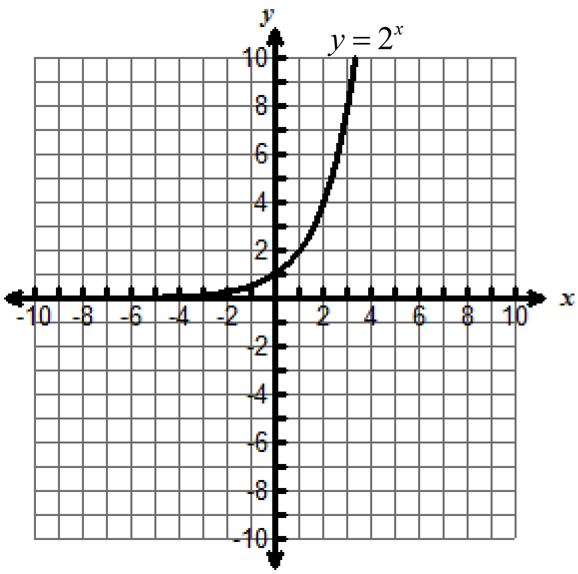
<u>Part 3</u>: Investigating Linear Functions
Notes to Leader: At the end of Part 3, participants should be able to determine the inverse of a linear function and describe an inverse relation as both a set of inverse number operations and a set of inverse transformations.

It is suggested that all participants do Part 3 (linear functions) together. Then, participants can jigsaw Part 4 (quadratic functions) and Part 5 (exponential functions) and discuss the similarities and differences among all three families of functions.

In Algebra 1, students investigate linear, quadratic, and exponential functions. The graphs of the parent functions are shown.

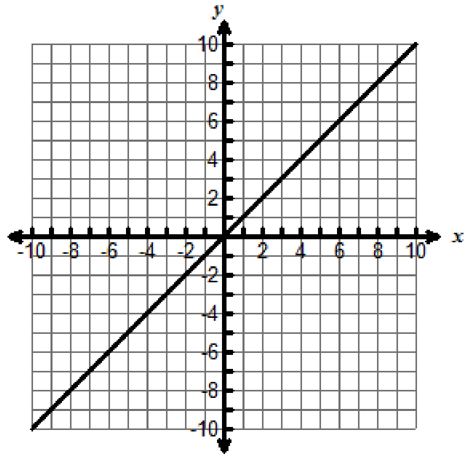






Trace each parent function onto a separate piece of patty paper. Be sure to trace and label the axes as well.

1. **Reflect the linear parent function across the line $y = x$. Sketch your resulting graph.**



2. **What is the domain and range of the inverse of the linear parent function? How do they compare with the original function?**
   The domain and range are both all real numbers, just like the original parent function.

3. **Is the inverse of the linear parent function also a function? How do you know?**
   Yes, the inverse is also a function since for each x-value there is only one corresponding y-value.

4. **What kind of function is the inverse of a linear function?**
   The inverse is also a linear function.

5. **What is the inverse of the function $y = \frac{2}{5}x - 7$? Find the inverse using at least two different methods.**
   $y = \frac{5}{2}(x+7)$ *or* $y = \frac{5}{2}x + \frac{35}{2}$

6. **How did you determine the inverse?**
   Participants could reflect the graph of the function across the line y = x then determine the equation of the reflected line.

   Symbolically, participants may reverse the domain (x) and range (y) then solve for y using inverse operations.

7. **What concepts and procedures did you apply to determine the inverse?**
   The concept of "inverse of a function," where the domain and range are reversed is applied.

   The procedures applied will vary depending on how participants found the inverse. Graphically, reflecting across the line y = x and determining the equation of a line from a *graph could be applied. Symbolically, applying inverse operations (in this case, addition then division) to solve for y could be applied.*

8. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{2}{5}x - 7$?

Multiply by $\frac{2}{5}$

Subtract 7

9. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{2}{5}x - 7$?

Add 7

Divide by $\frac{2}{5}$

Facilitation Question

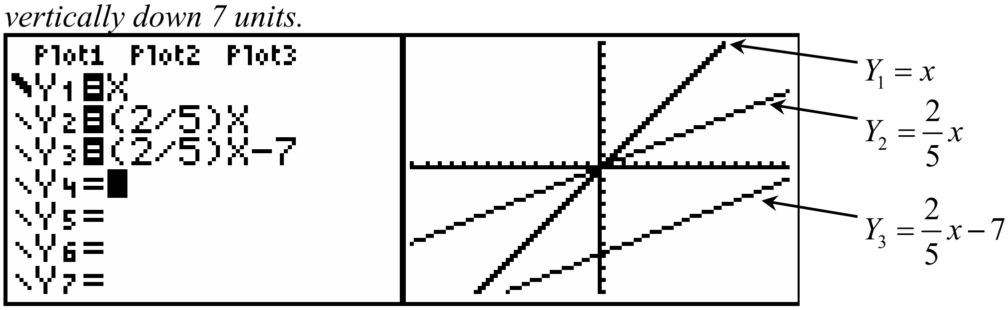
- How are multiplication and division of fractions related?
  Division by a fraction is symbolically equivalent to multiplying by the reciprocal.

10. How do these two sets of operations compare?

The operations done to the inverse function are the inverse operations from what is done to the original function. They are also done in reverse order.

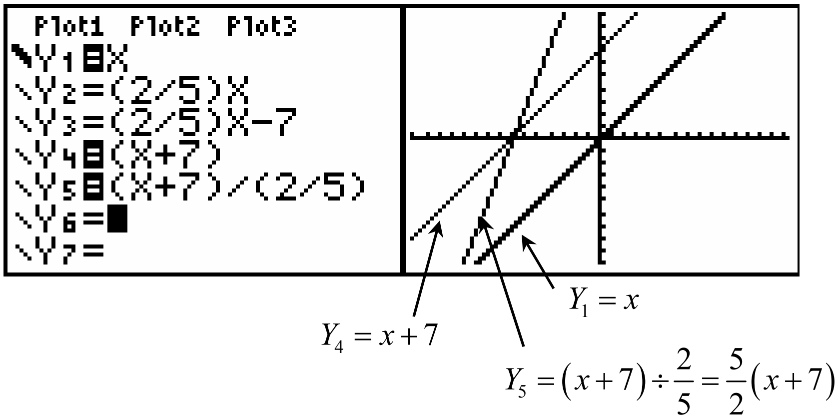
11. Describe the graph of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent function.

The parent function, y = x, is compressed vertically by a factor of $\frac{2}{5}$ *then translated vertically down 7 units.*



12. Describe the graph of the inverse of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent function.

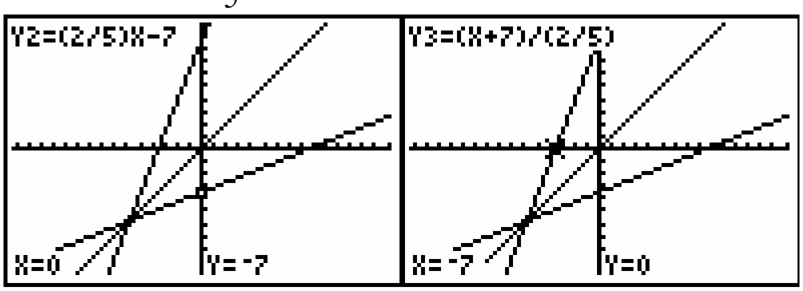
The parent function is shifted 7 units to the left then stretched vertically by a scale factor of $\frac{5}{2}$.



13. Compare the graphs of $y = \frac{2}{5}x - 7$, its inverse, and the line $y = x$.

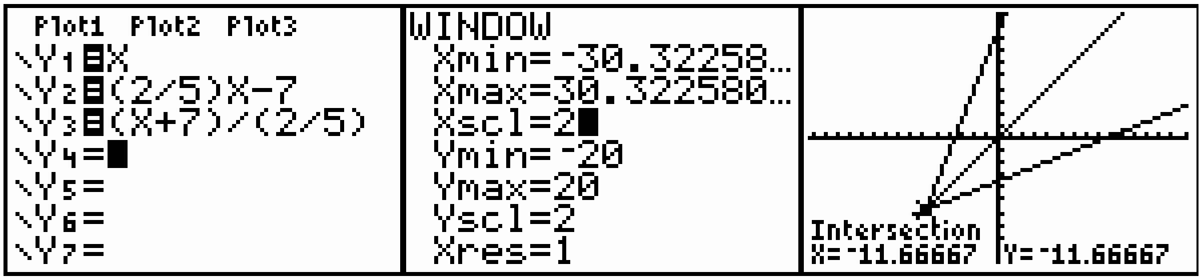
a. What do you notice about the intercepts of the graphs?

The y-intercept of $y = \frac{2}{5}x - 7$ is the same as the x-intercept of its inverse. Likewise, the x-intercept of $y = \frac{2}{5}x - 7$ is the same as the y-intercept of its inverse.



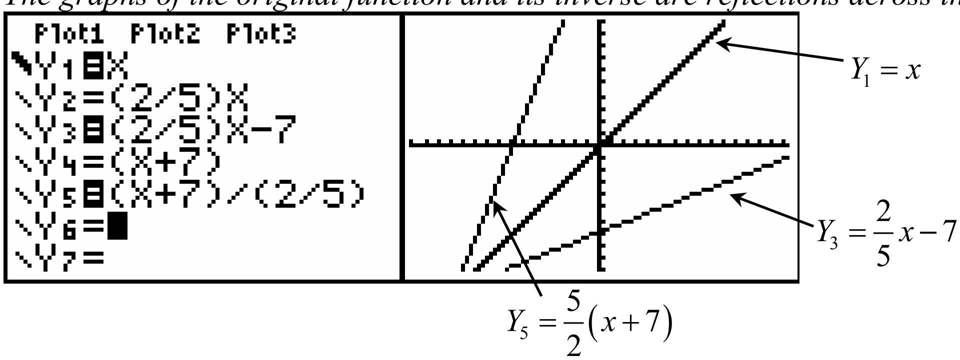
b. Where are the three graphs concurrent? What is the significance of this point?

The three graphs are concurrent at $\left(-11\frac{2}{3}, -11\frac{2}{3}\right)$. This is the point where the graph of the original function intersects its inverse. This point is also located on the line of reflection.



c. In terms of transformations, how do the graphs of the original function and its inverse compare?

The graphs of the original function and its inverse are reflections across the line y = x.



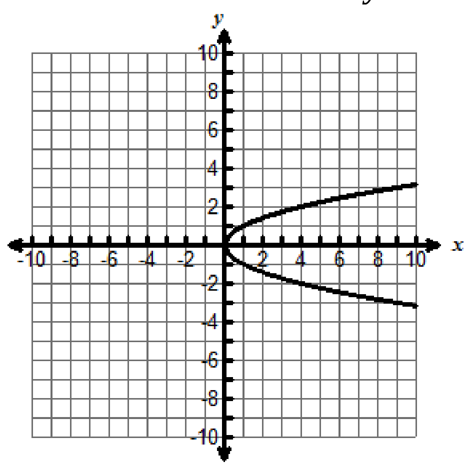
Also, transformations to the inverse relation are the inverse of the transformations (reverse transformations in reverse order) done to the original relation.

Note to Leader: Before moving on to Parts 4 and 5, be sure to discuss participants' responses to the last question. Draw out the common understandings of inverses as a set of inverse operations and inverse transformations.

<u>Part 4</u>: Quadratic Functions
Notes to Leader: At the end of Part 4, participants should be able to determine the inverse of a quadratic function (including restricting the range to result in a function) and connect inverse relations from a number operations perspective, a transformational perspective, and a symbolic perspective.

After doing Part 3 (linear functions) together, it is suggested that participants jigsaw Part 4 (quadratic functions) and Part 5 (exponential functions) then discuss the similarities and differences among all three families of functions.

1. **Reflect the quadratic parent function across the line *y* = *x*. Sketch your resulting graph.**



2. **What is the domain and range of the inverse of the quadratic parent function? How do they compare with the original function?**
   The domain is {x: x ≥ 0}, the range is all real numbers.
   The domain of the original parent function became the range of the inverse.
   The range of the original parent function became the domain of the inverse.

3. **Is the inverse of the quadratic parent function also a function? How do you know?**
   No. For most x-values (all except x = 0), there are two corresponding y-values. For example, when x = 4, y = +2 and −2.

4. **If the inverse is not a function, how can we restrict the domain and/or range of the original function so that the inverse is also a function?**
   If we only consider the domain {x: x ≥ 0}, then the inverse becomes a function.

5. **What kind of function is the inverse (range restricted) of a quadratic function?**
   The inverse of a quadratic function is a square root function.

6. Is either the original parent function or its inverse (range restricted) one-to-one? How do you know?
The parent function $y = x^2$ is not one-to-one since for all y-values except $y = 0$, there are two x-values yielding the same y-value. The inverse function (range restricted) is one-to-one since every y-value is only generated by one x-value.

7. What is the inverse function of the function $y = -3(x-1)^2 + 2$? Find the inverse using at least two different methods.
$y = 1 + \sqrt{\frac{x-2}{-3}}$ *or* $y = 1 - \sqrt{\frac{x-2}{-3}}$
Typically, the principal (positive) square root is used for square root functions.

8. How did you determine the inverse?
Graphically, participants could reflect the graph of the function across the line $y = x$ then determine the equation of the reflected curve.

Symbolically, participants may reverse the domain (x) and range (y) then solve for y using inverse operations.

9. What concepts and procedures did you apply to determine the inverse?
The concept of "inverse of a function," where the domain and range are reversed, is applied.

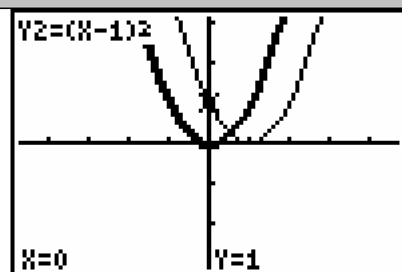
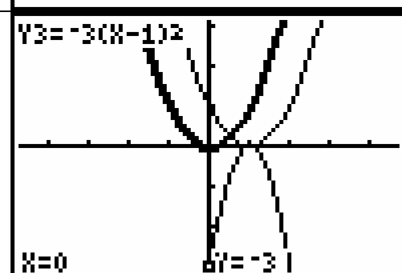
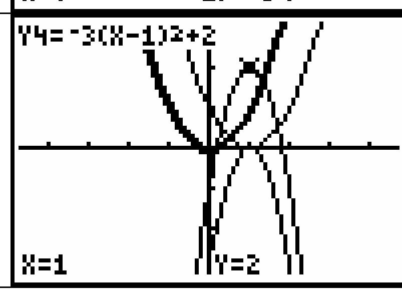
The procedures applied will vary depending on how participants found the inverse. Graphically, reflecting across the line $y = x$ and determining the equation of the curve from a graph could be applied. Symbolically, applying inverse operations (in this case, subtraction, division, square rooting, then addition) to solve for y could be applied.

10. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = -3(x-1)^2 + 2$?
Subtract 1
Square the value
Multiply by −3
Add 2

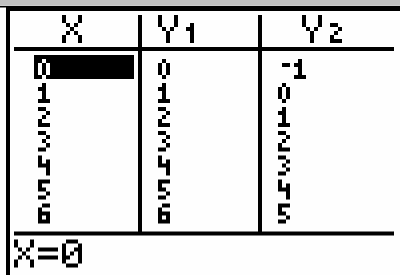
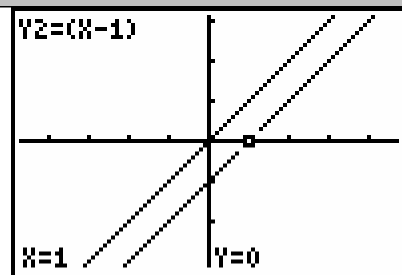
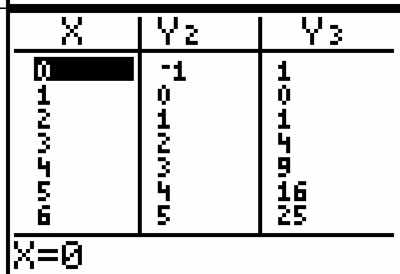
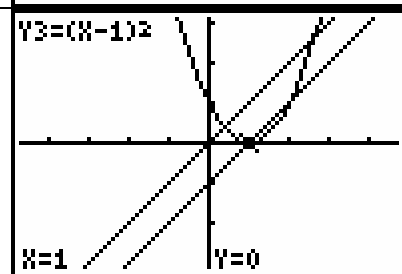
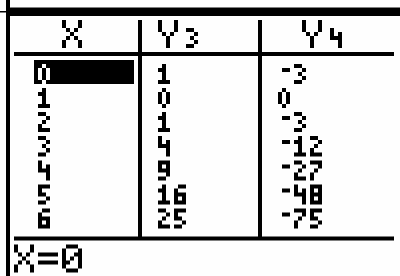
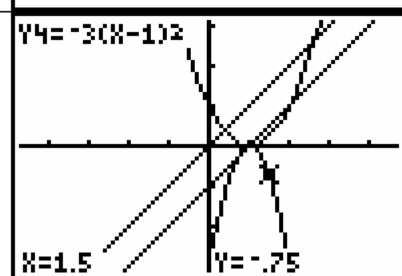
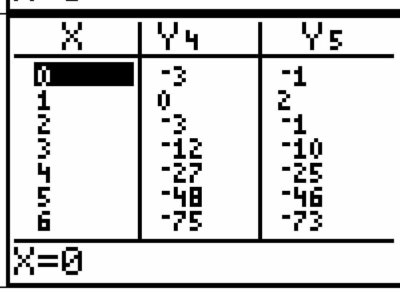
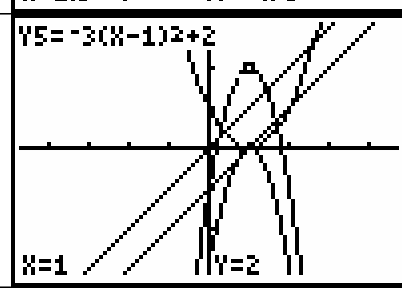
11. From the perspectives of number operations and transformations, develop the quadratic function, $y = -3(x-1)^2 + 2$, from the parent function $y = x^2$. Include tabular, graphical, and symbolic representations of the number operation as it applies to the function.

Note to Leader: *All graphs are shown using the viewing window at right.*

```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-3.1
 Ymax=3.1
 Yscl=1
 Xres=1
```

<table>
<tr><th>Number Operation</th><th>Tabular</th><th>Graphical</th><th>Symbolic</th></tr>
<tr><td>Subtract 1 from the x-value in the parent function $y = x^2$.</td><td><table><tr><th>X</th><th>Y1</th><th>Y2</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr><tr><td>2</td><td>4</td><td>1</td></tr><tr><td>3</td><td>9</td><td>4</td></tr><tr><td>4</td><td>16</td><td>9</td></tr><tr><td>5</td><td>25</td><td>16</td></tr><tr><td>6</td><td>36</td><td>25</td></tr><tr><td colspan="3">X=0</td></tr></table></td><td></td><td>$y = (x-1)^2$</td></tr>
<tr><td>Multiply by a factor of −3</td><td><table><tr><th>X</th><th>Y1</th><th>Y3</th></tr><tr><td>0</td><td>0</td><td>-3</td></tr><tr><td>1</td><td>1</td><td>0</td></tr><tr><td>2</td><td>4</td><td>-3</td></tr><tr><td>3</td><td>9</td><td>-12</td></tr><tr><td>4</td><td>16</td><td>-27</td></tr><tr><td>5</td><td>25</td><td>-48</td></tr><tr><td>6</td><td>36</td><td>-75</td></tr><tr><td colspan="3">X=0</td></tr></table></td><td></td><td>$y = -3(x-1)^2$</td></tr>
<tr><td>Add 2</td><td><table><tr><th>X</th><th>Y1</th><th>Y4</th></tr><tr><td>0</td><td>0</td><td>-1</td></tr><tr><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td><td>-1</td></tr><tr><td>3</td><td>9</td><td>-10</td></tr><tr><td>4</td><td>16</td><td>-25</td></tr><tr><td>5</td><td>25</td><td>-46</td></tr><tr><td>6</td><td>36</td><td>-73</td></tr><tr><td colspan="3">X=0</td></tr></table></td><td></td><td>$y = -3(x-1)^2 + 2$</td></tr>
</table>

12. Now, use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the quadratic function, $y = -3(x-1)^2 + 2$, from $y = x$.

Number Operation	Tabular	Graphical	Symbolic
Subtract 1 from the parent function $y = x$			$y = x - 1$
Square the value (apply the parent function operation)			$y = (x - 1)^2$
Multiply by a factor of -3			$y = -3(x - 1)^2$
Add 2			$y = -3(x - 1)^2 + 2$

13. How does each successive number operation transform the function numerically, graphically, and symbolically?

Responses may vary. Participants should connect the symbolic representation (general case) to the numeric representations contained in the table (specific cases).

Participants should also observe that the graphical representation is a set of points from the table that can be described generally by the symbolic representation. Each successive operation transforms the numeric/tabular values, and this transformation affects the symbolic (general rule) and graphical representations in turn.

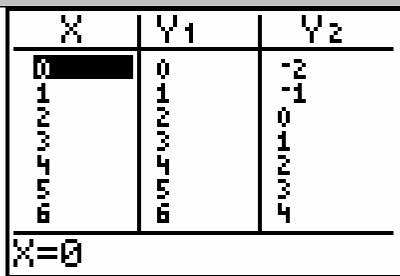
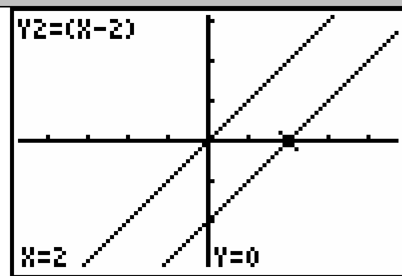
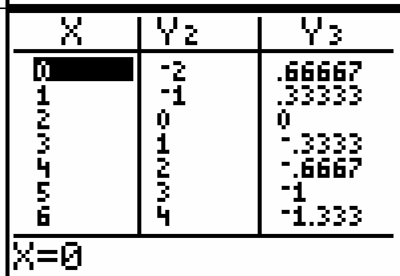
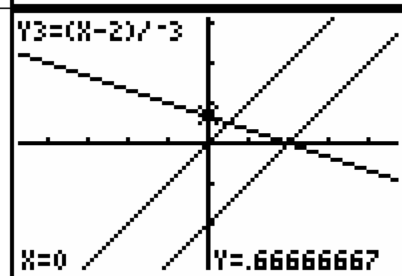
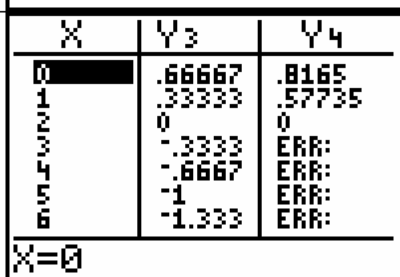
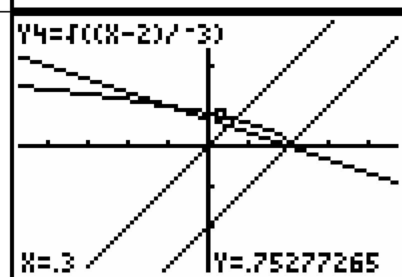
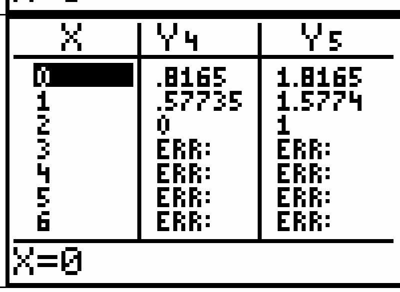
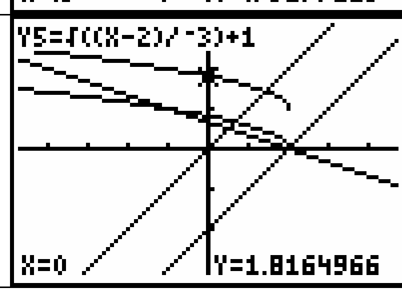
14. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = -3(x-1)^2 + 2$?

Subtract 2
Divide by −3
Take the square root
Add 1

15. How does this set of operations compare to the operations applied to generate the function $y = -3(x-1)^2 + 2$?

The operations done to the inverse function are the inverse operations from what is done to the original function. They are also done in reverse order.

16. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the square root inverse of the function, $y = -3(x-1)^2 + 2$, from $y = x$.

Number Operation	Tabular	Graphical	Symbolic
Subtract 2			$y = x - 2$
Divide by −3			$y = \frac{x-2}{-3}$ *or* $y = -\frac{1}{3}(x-2)$
Take the square root			$y = \sqrt{\frac{x-2}{-3}}$ *or* $y = \sqrt{-\frac{1}{3}(x-2)}$
Add 1			$y = \sqrt{\frac{x-2}{-3}} + 1$ *or* $y = 1 + \sqrt{-\frac{1}{3}(x-2)}$

17. How do the inverse operations relate to the inverse function?

The inverse function is a set of inverse operations performed in reverse order.

18. Generalize how inverse relations for quadratic functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.
Possible responses:

Number Operation	Tabular	Graphical	Symbolic
The number operations for inverse relations are the inverse operations that are applied to the original function, applied in reverse order.	*Range values from the table of the inverse relation match the domain values for the original function.*	*The graphs of a function and its inverse relation or function are reflections across the line $y = x$.*	*The inverse relation can be generated by reversing the domain and range elements (represented by x and y, respectively) then solving for y.*

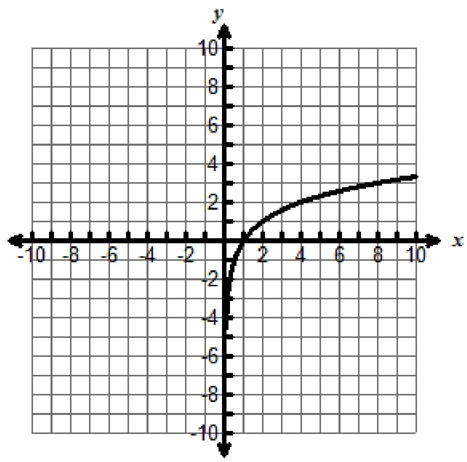
Note to Leader: When both groups come back together to debrief the jigsawed Parts 4 and 5, be sure to ask participants to discuss their responses to the last question of both parts.

Facilitation questions for the whole-group debrief of Parts 4 and 5 are at the end of Part 5.

<u>Part 5</u>: Exponential Functions
Notes to Leader: At the end of Part 5, participants should be able to determine the inverse of an exponential function (including restricting the range to result in a function) and connect inverse relations from a number operations perspective, a transformational perspective, and a symbolic perspective.

After doing Part 3 (linear functions) together, it is suggested that participants jigsaw Part 4 (quadratic functions) and Part 5 (exponential functions) then discuss the similarities and differences among all three families of functions.

1. **Reflect the exponential parent function, $y = 2^x$, across the line $y = x$. Sketch your resulting graph.**



2. **What is the domain and range of $y = 2^x$?**
   The domain is all real numbers and the range is {y: y > 0}.

3. **What is the domain and range of the inverse of $y = 2^x$? How do they compare with the original function?**
   The domain is {x: x > 0} and the range is all real numbers.
   The domain of the original parent function became the range of the inverse.
   The range of the original parent function became the domain of the inverse.

4. **What asymptote(s) does the original function, $y = 2^x$ have? Why does this asymptote exist?**
   There is a horizontal asymptote at y = 0 (the x-axis) because there are no powers of 2 that yield a negative value.

5. **What asymptote(s) does the inverse of $y = 2^x$ have? How do they compare to the asymptotes of the original function?**
   There is a vertical asymptote at x = 0 (the y-axis). This asymptote is a reflection of the original asymptote, y = 0, across the line y = x.

6. Is the inverse of $y = 2^x$ also a function? How do you know?
Yes, the inverse is also a function since for each x-value there is only one corresponding y-value.

7. What kind of function is the inverse of an exponential function?
The inverse of an exponential function is a logarithmic function.

8. Is either the original parent function or its inverse one-to-one? How do you know?
Both the parent function and the inverse function are one-to-one since every y-value is only generated by one x-value.

9. What is the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$? Find the inverse using at least two different methods.
$y = log\left(2\left(x-3\right)\right)+1$ *or* $y = log\left(2x-6\right)+1$

10. How did you determine the inverse?
Participants could reflect the graph of the function across the line y = x then determine the equation of the reflected curve.

Symbolically, participants may reverse the domain (x) and range (y) then solve for y using inverse operations.

11. What concepts and procedures did you apply to determine the inverse?
The concept of "inverse," where the domain and range are reversed is applied.

The procedures applied will vary depending on how participants found the inverse. Graphically, reflecting across the line y = x and determining the equation of the curve from a graph could be applied. Symbolically, applying inverse operations (in this case, subtraction, division, taking the logarithm, then addition) to solve for y could be applied.

12. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{1}{2}(10)^{x-1} + 3$?

Subtract 1
Raise 10 to that power
Multiply by $\frac{1}{2}$
Add 3

13. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$?

Subtract 3

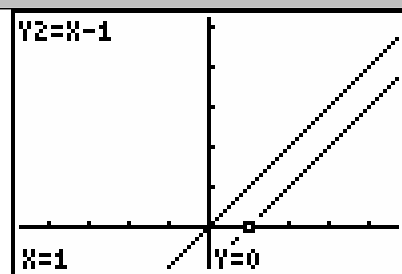
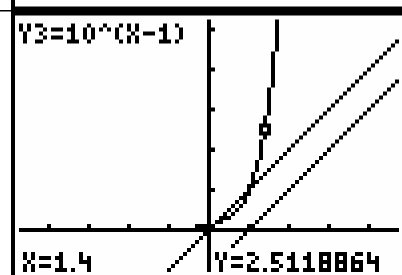
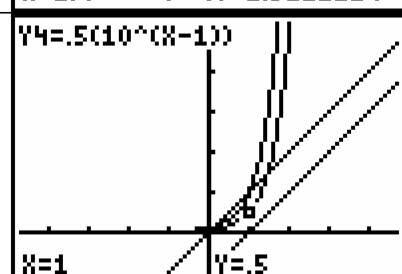
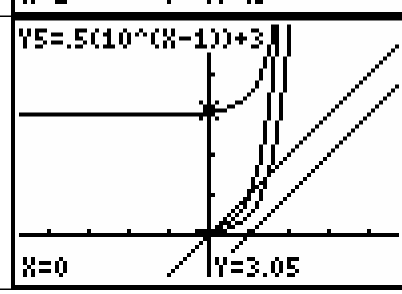
Multiply by 2 (divide by $\frac{1}{2}$)
Take the base-10 logarithm
Add 1

14. How do these two sets of operations compare?
The operations done to the inverse function are the inverse operations from what is done to the original function. They are also done in reverse order.

15. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the exponential function, $y=\frac{1}{2}(10)^{x-1}+3$, from $y = x$.

Note to Leader: *All graphs are shown using the following viewing window:*

```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-1
 Ymax=5.2
 Yscl=1
 Xres=█
```

Number Operation	Tabular	Graphical	Symbolic
Subtract 1	X, Y1, Y2 -3, -3, -4 -2, -2, -3 -1, -1, -2 0, 0, -1 1, 1, 0 2, 2, 1 3, 3, 2 X= -3		$y=x-1$
Raise 10 to that power	X, Y2, Y3 -3, -4, 1E-4 -2, -3, .001 -1, -2, .01 0, -1, .1 1, 0, 1 2, 1, 10 3, 2, 100 X=0		$y=10^{x-1}$
Multiply by $\frac{1}{2}$	X, Y3, Y4 -3, 1E-4, 5E-5 -2, .001, 5E-4 -1, .01, .005 0, .1, .05 1, 1, .5 2, 10, 5 3, 100, 50 X=0		$y=\frac{1}{2}\left(10^{x-1}\right)$
Add 3	X, Y4, Y5 -3, 5E-5, 3.0001 -2, 5E-4, 3.0005 -1, .005, 3.005 0, .05, 3.05 1, .5, 3.5 2, 5, 8 3, 50, 53 X=0		$y=\frac{1}{2}\left(10^{x-1}\right)+3$

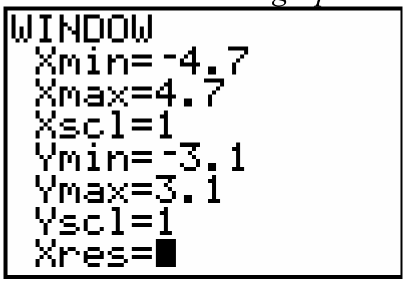
16. How does each successive number operation transform the function numerically, graphically, and symbolically?

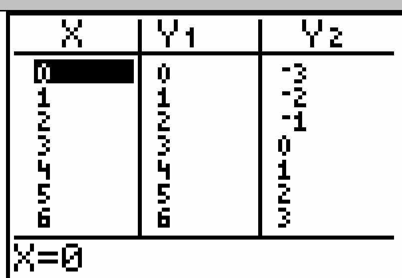
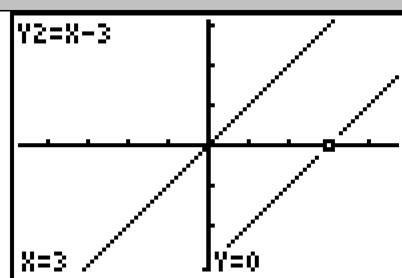
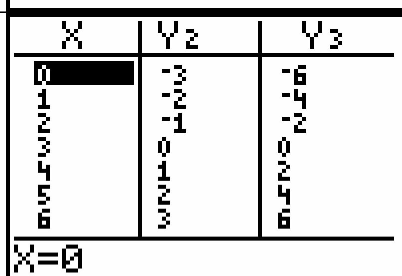
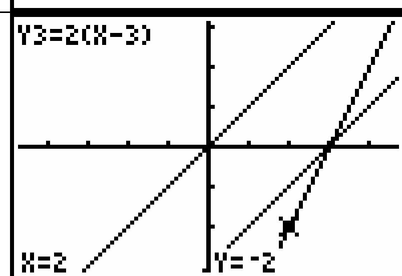
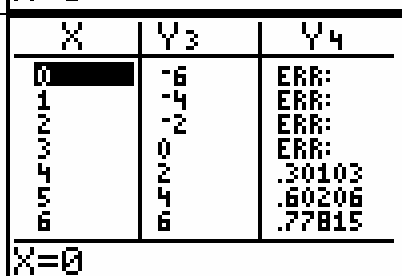
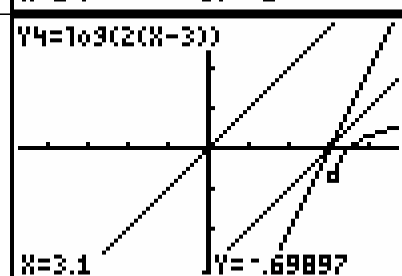
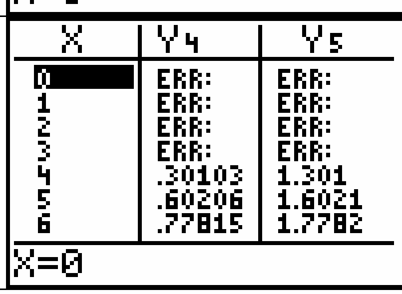
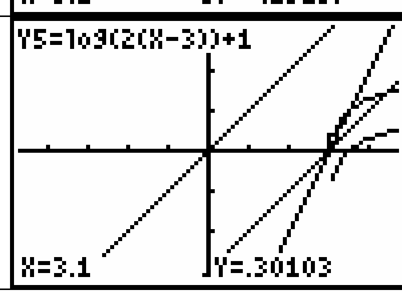
Responses may vary. Participants should connect the symbolic representation (general case) to the numeric representations contained in the table (specific cases).

Participants should also observe that the graphical representation is a set of points from the table that can be described generally by the symbolic representation. Each successive operation transforms the numeric/tabular values and this transformation affects the symbolic (general rule) and graphical representations in turn.

17. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the logarithm inverse of the function, $y = \frac{1}{2}(10)^{x-1} + 3$, from $y = x$.

Note to Leader: *All graphs are shown using the following viewing window:*



Number Operation	Tabular	Graphical	Symbolic
Subtract 3			$y = x - 3$
Multiply by 2 (divide by $\frac{1}{2}$)			$y = 2(x-3)$
Take the base-10 logarithm			$y = log(2(x-3))$
Add 1			$y = log(2(x-3)) + 1$

18. How do the inverse operations relate to the inverse function?
The inverse function is a set of inverse operations performed in reverse order.

19. Generalize how inverse relations for exponential functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.
Possible responses:

Number Operation	Tabular	Graphical	Symbolic
The number operations for inverse relations are the inverse operations that are applied to the original function, applied in reverse order.	*Range values from the table of the inverse relation match the domain values for the original function.*	*The graphs of a function and its inverse relation or function are reflections across the line y = x.*	*The inverse relation can be generated by reversing the domain and range elements (represented by x and y, respectively) then solving for y.*

Use the Facilitation Questions to debrief the experiences in Parts 3, 4, and 5.

Facilitation Questions

- How are inverses of linear, quadratic, and exponential functions similar?
  The inverses are all reflections across the line $y = x$.
  Transformations of the parent function result in similar translations and vertical dilations.
  Attributes of x become attributes of y; e.g., x-intercepts of the original function become y-intercepts of the inverse relation and vice versa.

- How are inverses of linear, quadratic, and exponential functions different?
  Inverses of linear and exponential functions are functions while inverses of quadratic functions are not functions unless the range is restricted.

- How do different limitations on domain and range (maximum/minimum value, asymptotes, discontinuities, etc.) affect inverse relations/functions?
  A maximum/minimum value such as the vertex of a quadratic function results in a function that is not one-to-one. Inverses of these functions are non-functional relations. Asymptotes such as the horizontal asymptote in an exponential function are also reflected across the line $y = x$ when the logarithmic inverse function is generated.

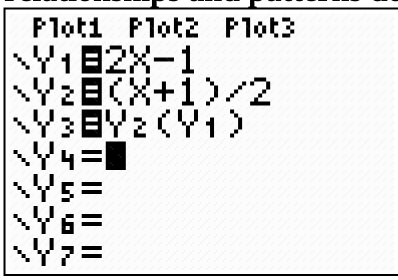
- What kinds of functions have inverses that are not functions without domain/range restrictions?
  Functions that are not one-to-one such as quadratic functions (or participants may mention absolute value or other functions depending on their prior experiences) have inverses that are not functions without domain/range restrictions.

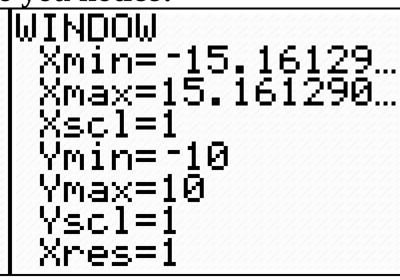
- What kinds of functions have inverses that are functions without the need to restrict the domain/range?
  One-to-one functions such as linear or exponential have inverses without the need for domain/range restrictions.

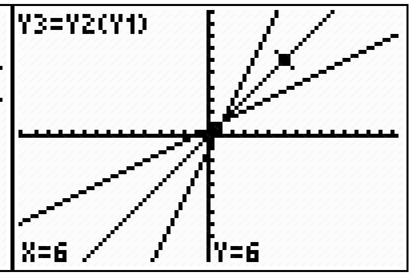
<u>Part 6</u>: Compositions of Functions (Extension)
Note to Leader: At the end of Part 6, participants should recognize and describe attributes of composition of functions, including whether or not composition is commutative and how to account for any domain/range restrictions on the function or its inverse.

Enter the function *y* = 2*x* – 1 into *Y1* of your graphing calculator's function editor. Enter the inverse of this function into *Y2*. Enter the composition of *Y2* and *Y1* into *Y3*.

1. **Sketch the graphs of the three functions (describe your viewing window). What relationships and patterns do you notice?**



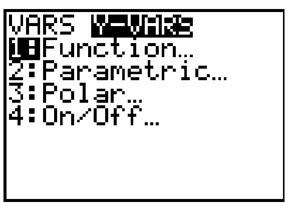




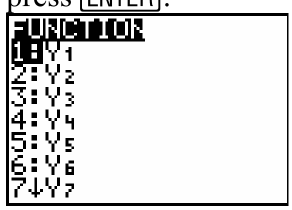
Participants may observe that the composition of the two functions is the line y = x.

Technology Facilitation Tip:
To find the y-variables as shown in the first screen shot above:

Press [VARS] to select the Variables menu. Use the [▸] arrow key to select *y*-variables (Y-VARS).



Press [1] to select FUNCTION variables. Use the [▴][▾] arrow keys to select the desired y-variable then press [ENTER].



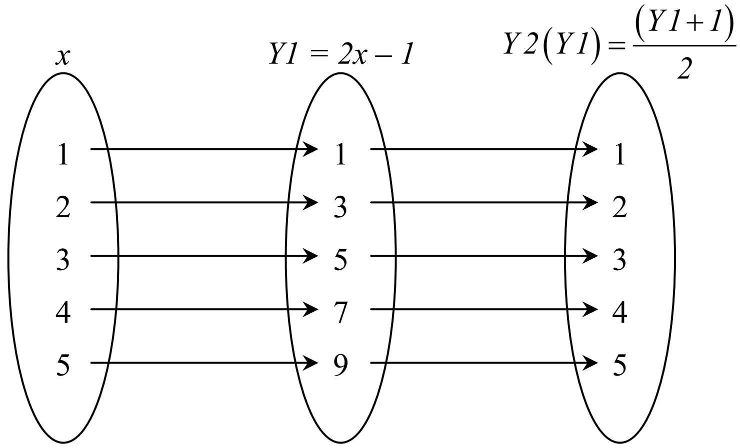
Function notation can be used in the function editor to select an input other than X. For example, to find Y2 values for a domain (input values) of Y1, type Y2(Y1) as shown in the first screen shot above.

2. **Look at the table values for each of the three functions. What do you notice?**
Responses may vary. Participants should notice that the values for Y3 are the same as the x-values.

X	Y2	Y3
1	1	1
2	1.5	2
3	2	3
4	2.5	4
5	3	5
6	3.5	6
7	4	7

Y3=1

3. How could you represent the composition of these two functions with a mapping?



4. What effect does composition of inverse functions have? Why do you think this is so?
Composition of inverse functions maps all domain values back onto themselves. This works because the first function performs arithmetic operations on the domain values then the inverse function "undoes" the arithmetic operations.

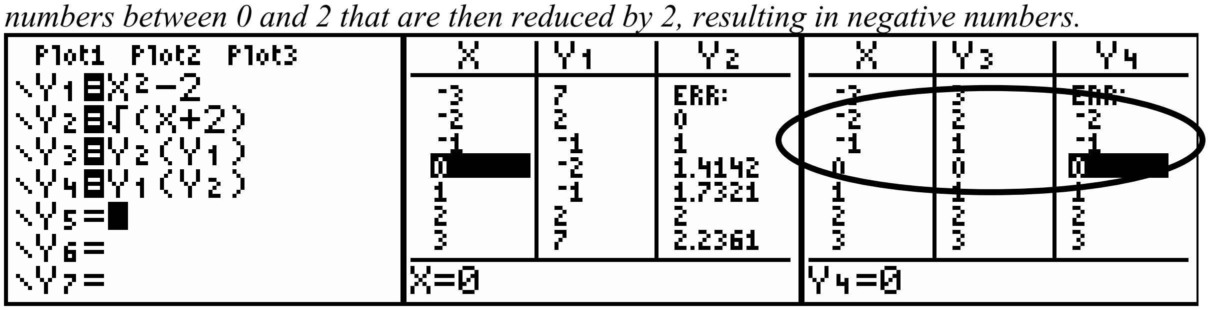
5. Does the order of composition matter? Explain your answer.
For linear functions and their inverses, the order does not matter because there are no domain or range restrictions.

Note to Leader: However, if two inverses have domain or range restrictions (such as the case with quadratic functions and their inverse square-root functions which will be investigated momentarily), the compositions could yield different subsets of real numbers, depending on which function operated on the set of real numbers first.

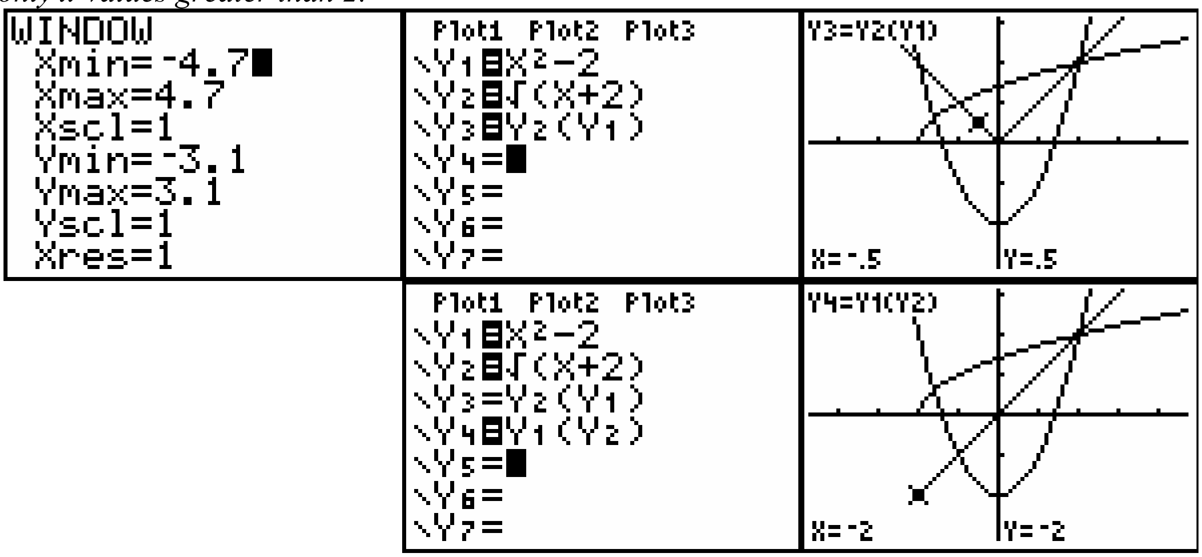
After participants complete the next prompt, be sure to ask them to discuss their findings.

6. Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse <u>function</u>. Describe the result numerically, graphically, and symbolically.

Numerically, neither composition returns the original x-values. Further, the order of composition yields a different set of y-values for $\{x : -2 \le x < 0\}$. Over this domain, the composition $Y2 \circ Y1$ yields the square root of a collection of positive numbers. Thus, the result is always positive. Meanwhile, the composition $Y1 \circ Y2$ yields a collection of positive numbers between 0 and 2 that are then reduced by 2, resulting in negative numbers.



Graphically, the composition of $Y2 \circ Y1$ yields the graph of the absolute value parent function, $y = |x|$. The composition of $Y1 \circ Y2$ yields the linear parent function, y = x, restricted to the domain x ≥ 2. Interestingly, the domain restriction corresponds to the range restriction on $Y2 = \sqrt{x+2}$ because the range of Y2 becomes the domain of Y1, consisting of only x-values greater than 2.



Symbolically, $Y1 = f(x) = x^2 - 2$, $Y2 = g(x) = \sqrt{x+2}$, $Y3 = Y2 \circ Y1 = g \circ f(x)$, and $Y4 = Y1 \circ Y2 = f \circ g(x)$. The composition functions reduce to:

$$\begin{aligned} g \circ f(x) &= \sqrt{(x^2 - 2) + 2} \\ &= \sqrt{x^2} \\ &= |x| \end{aligned} \qquad \begin{aligned} f \circ g(x) &= \left(\sqrt{x+2}\right)^2 - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Note to Leader: Discuss participants' responses before proceeding to Part 7.

Facilitation Questions

- What do the different representations of the composition of a quadratic function and its inverse function reveal? How are those revelations related?
  The tabular/numerical representations show specific cases of input-output for both compositions. The graphical representations spatially show how the two compositions are related to each other and to the original and inverse functions. The symbolic representations show the subtle yet important differences between the two compositions $f \circ g$ and $g \circ f$.

- Based on your experiences with quadratic functions, what would you expect compositions of other types of functions and their inverses to be like? Why?
  Responses may vary. Participants may conjecture that exponential functions would behave more like linear functions in that they are both continuously increasing or decreasing functions.

<u>Part 7</u>: Extension of Compositions

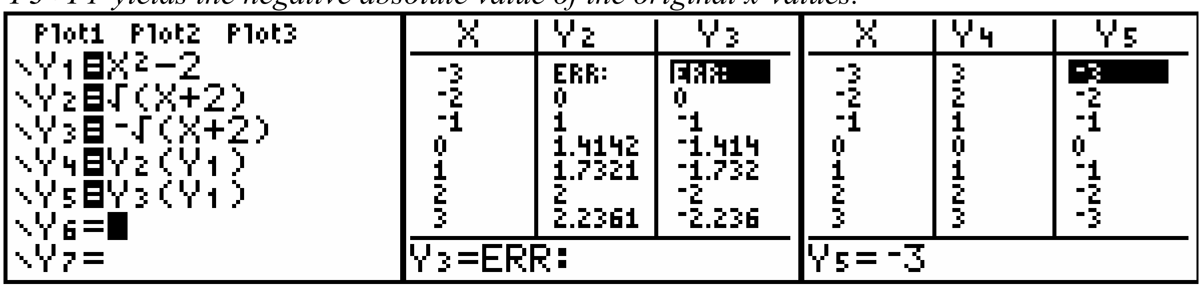
Note to Leader: Part 7 is an optional extension of compositions from Part 6. It is designed to allow participants to extend their understanding of compositions of inverse relations and functions. If participants do not experience Part 7, you can still have a meaningful discussion during the Explain and Elaborate phases.

1. **Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse <u>relation</u>. Describe the result numerically, graphically, and symbolically.**

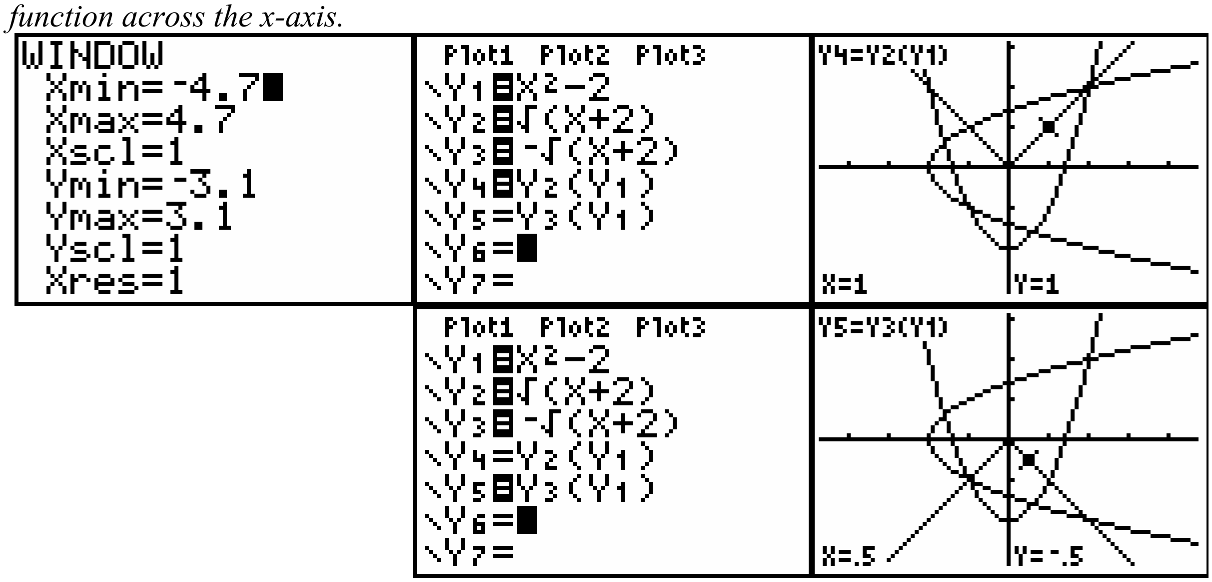
   Note to Leader: *Because the inverse <u>relation</u> of $y = x^2 - 2$ is a parabola opening up along the positive x-axis, we must plot the positive square root and the negative square root as two separate functions. To make screen shots and graphical analysis easier, we will consider $g \circ f$ first then explore $f \circ g$.*

   Case 1: Inverse is not a function – Composition of Inverse(Original)

   Numerically, the composition $Y2 \circ Y1$ yields the absolute value of the original x-values and $Y3 \circ Y1$ yields the negative absolute value of the original x-values.



Graphically, the composition of $Y2 \circ Y1$ yields the graph of the absolute value parent function, $y = |x|$. The composition of $Y3 \circ Y1$ yields a reflection of the absolute value parent function across the x-axis.

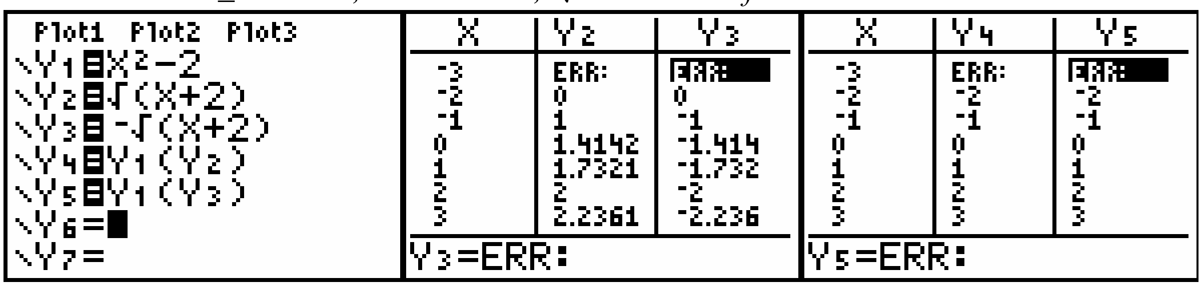


Symbolically, $f(x) = x^2 - 2$ *and* $g(x) = \pm\sqrt{x+2}$, *which is not a function. The composition* $g \circ f(x)$ *reduces to:*

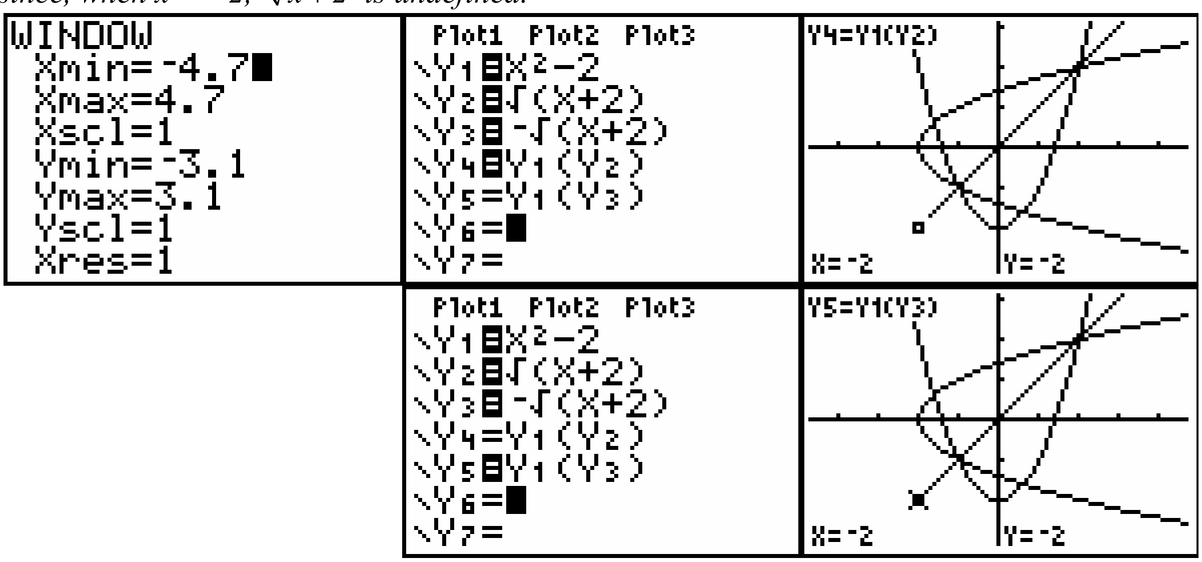
$$\begin{aligned} g \circ f(x) &= \pm\sqrt{(x^2 - 2) + 2} \\ &= \pm\sqrt{x^2} \\ &= \pm|x| \end{aligned}$$

Case 2: Inverse is not a function – Composition of Original(Inverse)

Numerically, both compositions $Y1 \circ Y2$ *and* $Y1 \circ Y3$ *yield the original x-values only for the domain where* $x \geq -2$ *since, when* $x < -2$, $\sqrt{x+2}$ *is undefined.*



Graphically, both compositions yield the line y = x with the domain restriction of $x \geq -2$ *since, when* $x < -2$, $\sqrt{x+2}$ *is undefined.*



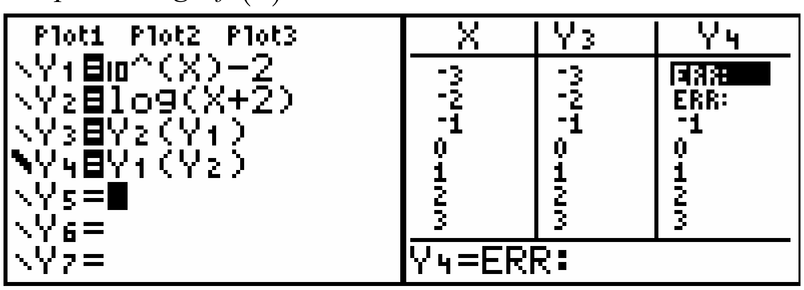
Symbolically, $f(x) = x^2 - 2$ *and* $g(x) = \pm\sqrt{x+2}$*, which is not a function. The composition* $f \circ g(x)$ *reduces to y = x as shown. However, due to range restrictions on the original square root functions (recall that the range of the first function becomes the domain of the second function), there is a limited domain to be operated on by f(x), resulting in a domain restriction on the composed function* $f \circ g(x)$.

$$\begin{aligned} f \circ g(x) &= \left(\pm\sqrt{x+2}\right)^2 - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

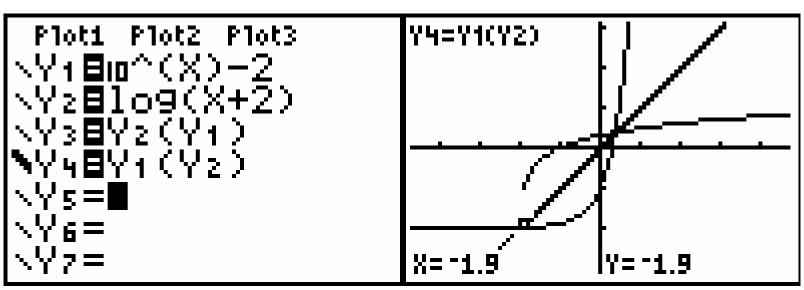
2. **Based on your experiences with linear and quadratic functions, what would you expect to be true about compositions of other types of functions, such as exponential, rational, or polynomial functions? Give examples or counterexamples.**
Responses may vary. Participants may conjecture that for polynomial functions, patterns exist among even and odd functions. Or, participants may conjecture that since exponential/logarithmic functions exhibit domain restrictions, similar patterns of restrictions on compositions hold true.

For example, consider the function $f(x) = 10^x - 2$ *and its inverse* $g(x) = \log(x+2)$.

Numerically, the range of the composition $f \circ g(x)$ *is a subset of the range of the composition* $g \circ f(x)$.



Graphically, the graph of $f \circ g(x)$ *(bold line shown in the graph) overlaps the graph of* $g \circ f(x)$ *only when x > −2. Hence, the range of* $f \circ g(x)$ *is a subset of the range of* $g \circ f(x)$.



Symbolically: $f\left(x\right)=10^{x}-2$ *and* $g\left(x\right)=\log\left(x+2\right)$, *so:*

$$\begin{aligned} g\circ f\left(x\right) &= \log\left(\left(10^{x}-2\right)+2\right) \\ &= \log\left(10^{x}\right) \\ &= x\log 10 \\ &= x \end{aligned}$$

$$\begin{aligned} f\circ g\left(x\right) &= 10^{\log\left(x+2\right)}-2 \\ &= x+2-2 \\ &= x^{*} \end{aligned}$$

** Since the domain values for g(x) are restricted to x > −2, the range of* $f\circ g\left(x\right)$ *is also restricted to x > −2.*

Note to Leader: Ask participant volunteers to share out their examples and counterexamples. Discuss similarities and differences.

Explain

Leaders' Note: *The Maximizing Algebra II Potential (MAP) professional development is intended to be an extension of the ideas introduced in Mathematics TEKS Connections (MTC). Throughout the professional development experience, we allude to components of MTC such as the Processing Framework Model, the emphasis of making connections among representations, and the links between conceptual understanding and procedural fluency.*

Debriefing the Experience:

1. **What concepts did we explore in the previous set of activities? How were they connected?**
   Responses may vary. Participants should observe that inverses are relations whose domains and ranges are reversed. Composition of functions is a foundations for functions concept in that the range of one function is used as the domain for another.

2. **What procedures did we use to describe inverses of functions? How are they related?**
   Tabular, graphical, and symbolic procedures were all used throughout the Explore phase. Ultimately, they are all connected through the numerical relationships used to generate them.

3. **What knowledge from Algebra I do students bring about linear, quadratic, and exponential functions?**
   According to the Algebra I TEKS, students study linear functions in depth. In Algebra I, students transform quadratic functions vertically (stretch/compress, reflect vertically, and translate), analyze the graphs of quadratic functions, solve quadratic equations, and connect roots/zeroes/x-intercepts. In Algebra I, students model growth and decay using exponential functions.

4. **After working with inverses of relations and functions in Algebra II, what are students' next steps in Precalculus or other higher mathematics courses?**
   According to the Precalculus TEKS, students will expand their catalog of curves to include natural logarithms, power functions, and trigonometric functions. Also, students will perform compositions on functions and find inverses of functions, including trigonometric functions such as y = arcsin(x).

Anchoring the Experience:

5. **Distribute to each table group a poster-size copy of the Processing Framework Model.**

6. **Ask each group to respond to the question:**

   Where in the processing framework would you locate the different activities from the Explore phase?

7. **Participants can use one color of sticky notes to record their responses. In future Explain phases, participants will use other colors to record their responses.**

Horizontal Connections within the TEKS

8. **Direct the participants' attention to the second layer in the Processing Framework Model: Horizontal Connections among Strands.**

9. **Prompt the participants to study the Algebra II TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

10. **Invite each table group to share 2 connections that they found and record them so that they are visible to the entire group.**

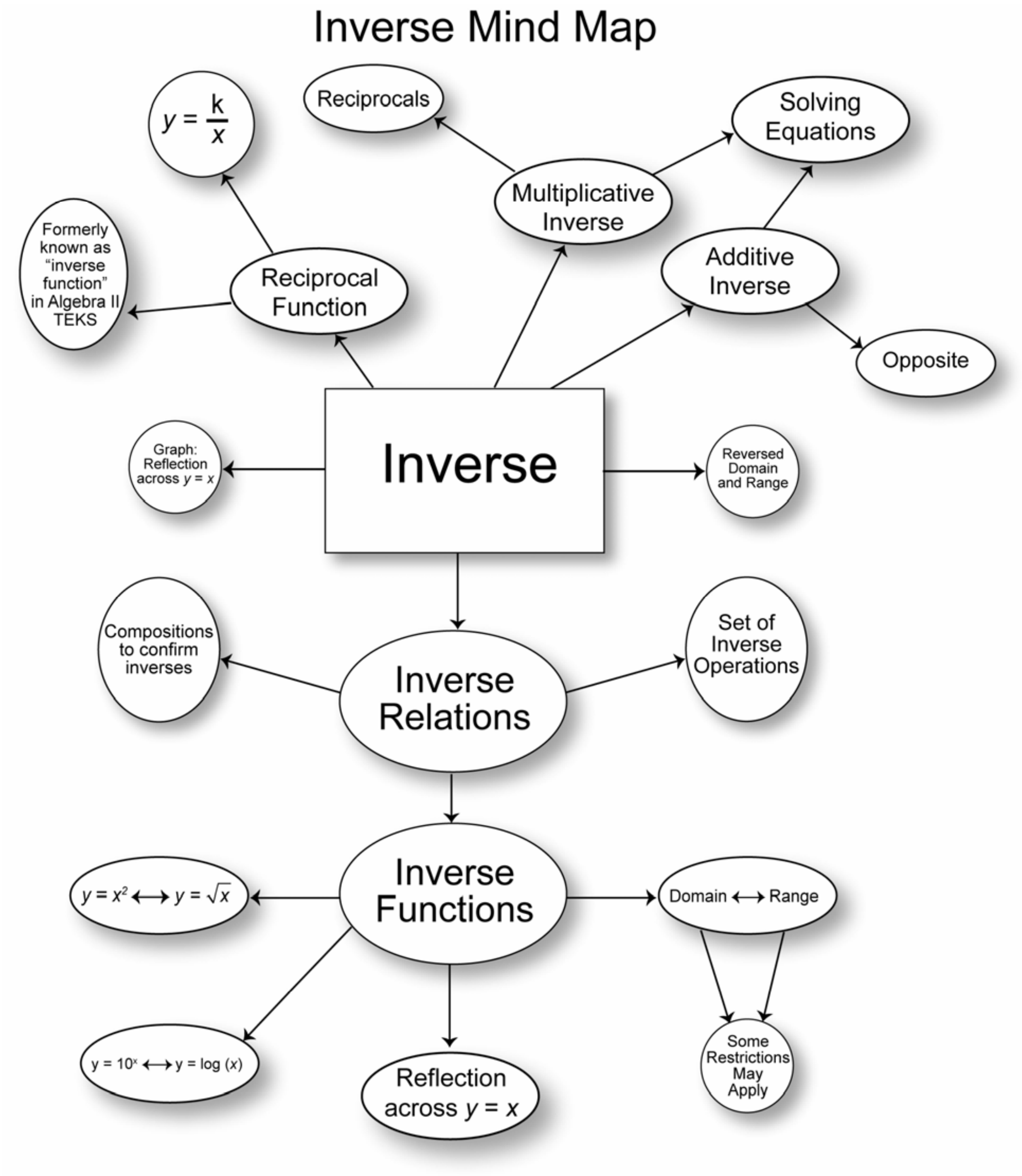
Vertical Connections within the TEKS

11. **Direct the participants' attention to the third layer in the Processing Framework Model: Vertical Connections across Grade Levels.**

12. **Prompt the participants to study the Algebra I, Geometry, Math Models, and Precalculus TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

13. **Invite each table group to share 2 connections that they found, recording so that the entire large group may see.**

14. **Provide each group of participants with a clean sheet of chart paper. Ask them to create a "mind map" for the mathematical term of "inverse."**
    Note to Leader: *If participants ask, be purposefully vague in not clarifying what you mean by "inverse." Part of the purpose of this activity is for participants to realize that students (especially English Language Learners, or ELLs) can be confused by the many meanings of the same term.*


Inverse Mind Map
$y = \frac{k}{x}$
Reciprocals
Solving Equations
Multiplicative Inverse
Formerly known as "inverse function" in Algebra II TEKS
Reciprocal Function
Additive Inverse
Opposite
Inverse
Graph: Reflection across $y = x$
Reversed Domain and Range
Compositions to confirm inverses
Inverse Relations
Set of Inverse Operations
Inverse Functions
$y = x^2 \longleftrightarrow y = \sqrt{x}$
Domain $\longleftrightarrow$ Range
$y = 10^x \longleftrightarrow y = \log(x)$
Reflection across $y = x$
Some Restrictions May Apply

Provide an opportunity for each group to share their mind maps with the larger group. Discuss similarities, differences, and key points brought forth by participants.

Facilitation Questions

- How do the different meanings of the term "inverse" relate to each other?
  Responses may vary. "Inverse" relations use sets of "inverse" operations. "Inverse" trigonometric functions (for example, arcsine) are inverses of the set of trigonometric functions.

- Where would compositions fit into your map?
  Responses may vary depending on participants' mind maps.

15. Distribute the Vocabulary Organizer template to each participant. Ask participants to construct a vocabulary model for the term "inverse relations/functions."
Possible response:

My definition

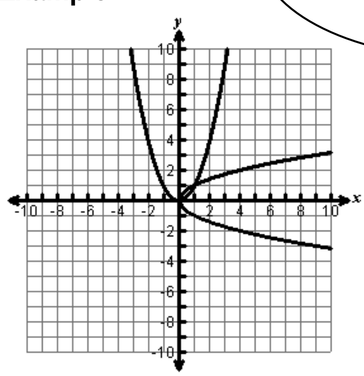
Relations whose domains and ranges are reversed.

Personal Association

Cell phone address book—When I dial, the names are the input and phone numbers are the output. But for incoming calls, the phone number is the input and the name is the output.

Inverse Relations and Functions

Example



Non-Example

$y = \frac{2}{3}x - 5$ *and* $y = \frac{3}{2}x + 5$ *are not inverse functions.*

16. When participants have completed their vocabulary models, ask participants to identify strategies from their experiences so far in the professional development that could be used to support students who typically struggle with Algebra II topics.

Note to Leader: *You may wish to have each small group brainstorm a few ideas first, then share their ideas with the large group while you record their responses on a transparency or chart paper.*

17. **How would this lesson maximize performance in Algebra II for teaching and learning the mathematical concepts and procedures associated with inverses of relations and functions?**

    Responses may vary. Anchoring procedures within a conceptual framework helps students understand what they are doing so that they become more fluent with the procedures required to accomplish their tasks. Problems present themselves in a variety of representations; providing students with multiple procedures to solve a given problem empowers students to solve the problem more easily.

Elaborate

Leaders' Note: *In this phase, participants will extend their learning experiences to their classroom.*

1. **Provide each participant with a copy of the 5E Student Lesson planning template. Ask participants to think back to their experiences in the Explore phase. Pose the following task:**

   What might a student-ready 5E lesson on inverses of relations/functions look like?
   - **What would the Engage look like?**
   - **Which experiences/activities would students explore firsthand?**
   - **How would students formalize and generalize their learning?**
   - **What would the Elaborate look like?**
   - **How would we evaluate student understanding of inverses of relations/functions?**

2. **After participants have recorded their thoughts, direct them to the student lesson for inverses of relations/functions. Allow time for participants to review lessons.**

3. **How does this 5E lesson compare to your vision of a student-centered 5E lesson?**
   Responses may vary.

4. **How does this lesson help remove obstacles that typically keep students from being successful in Algebra II?**
   By connecting symbolic manipulation to conceptual understanding as revealed in other representations (such as graphing), students have other tools with which to solve meaningful problems.

5. **How does this lesson maximize your instructional time and effort in teaching Algebra II?**
   Taking time to create a solid conceptual foundation reduces the need for re-teaching time and effort.

6. **How does this lesson maximize student learning in Algebra II?**
   Using multiple representations and foundations for functions concepts allows students to make connections among different ideas. These connections allow students to apply more quickly and readily their learning to new situations.

7. **How does this lesson accelerate student learning and increase the efficiency of learning?**
   Foundations for functions concepts such as function transformations transcend all kinds of functions. A basic toolkit for students to use when working with functions allows students to rethink what they know about linear and quadratic functions while they are learning concepts and procedures associated with other function families.

8. Read through the suggested strategies on Strategies that Support English Language Learners. Consider the possible strategies designed to increase the achievement of English language learners.

As participants read through the strategies that support English language learners and strategies that support students with special needs, they may notice that eight of the ten strategies are the same. The intention is to underscore effective teaching practices for all students. However, English language learners have needs specific to language that students with special needs may or may not have. The two strategies that are unique to the English language learners reflect an emphasis on language. Students with special needs may have prescribed modifications and accommodations that address materials and feedback. Students with special needs often benefit from progress monitoring with direct feedback and adaptation of materials for structure and/or pacing. A system of quick response is an intentional plan to gather data about a student's progress to determine whether or not the modification and (or) accommodation are (is) having the desired effect. The intention of the strategies is to provide access to rigorous mathematics and support students as they learn rigorous mathematics.

9. What evidence of these strategies do you find in this portion of the professional development?
Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the expected teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening, contributing to the teacher's ability to create an emotionally safe environment for learning.*

10. Which strategies require adaptation of the materials in this portion of the professional development?
Responses may vary. Most of the strategies are incorporated throughout the materials.

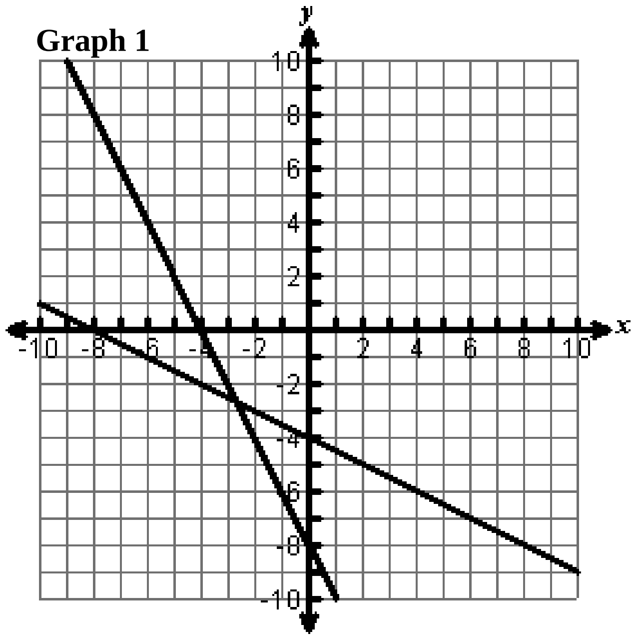
11. Read through the suggested strategies on Strategies that Support Students with Special Needs. Consider the possible strategies designed to increase the achievement of students with special needs.

12. What evidence of these strategies do you find in this portion of the professional development?
Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening, contributing to the teacher's ability to create an emotionally safe environment for learning.*

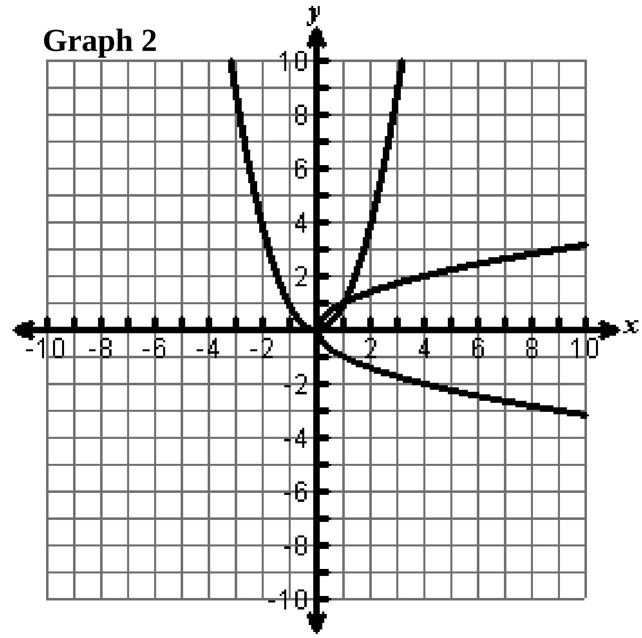
13. Which strategies require adaptation of the materials in this portion of the professional development?
Responses may vary. Most of the strategies are incorporated throughout the materials. Some materials may need to be modified for format.

Know When to Fold 'Em

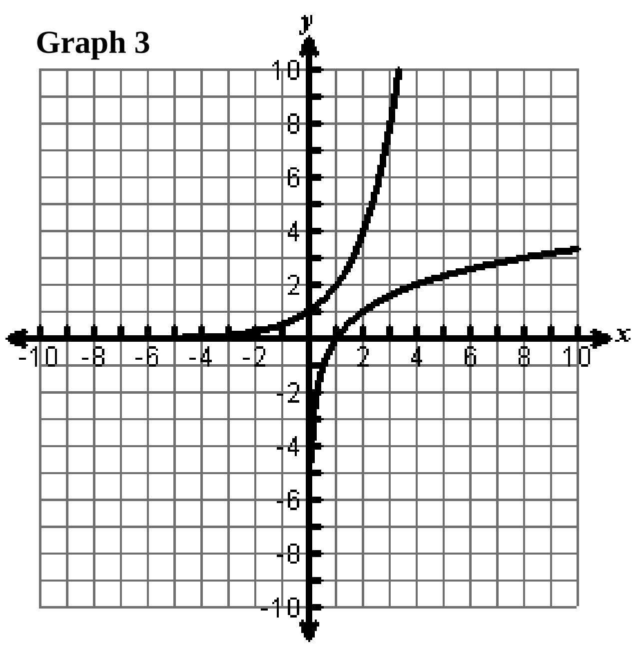
Graph 1



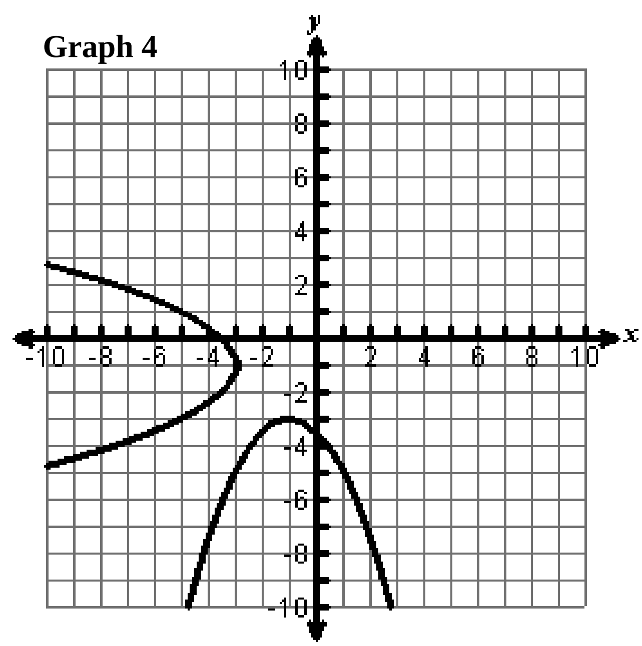
Graph 2



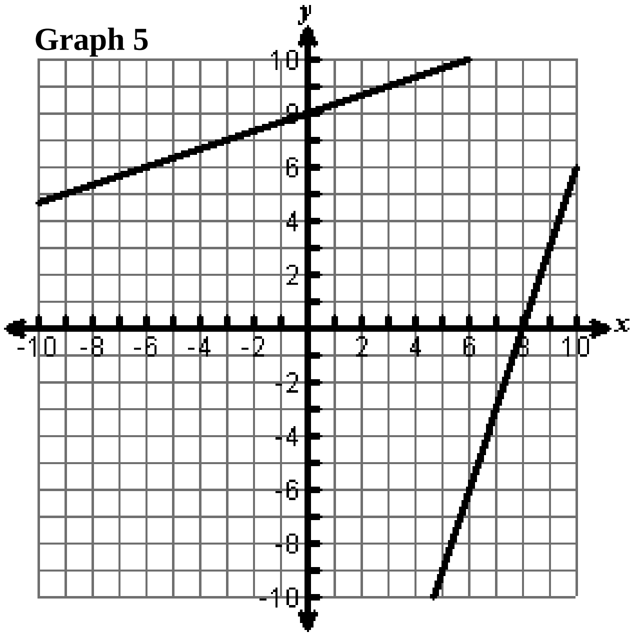
Graph 3



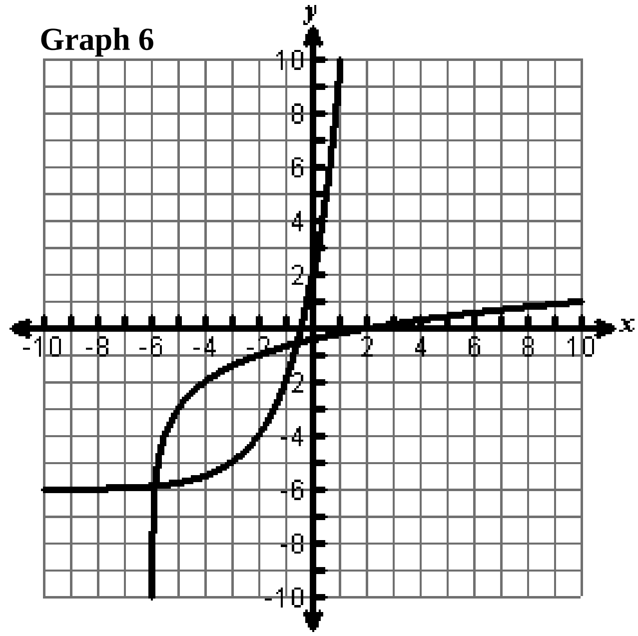
Graph 4



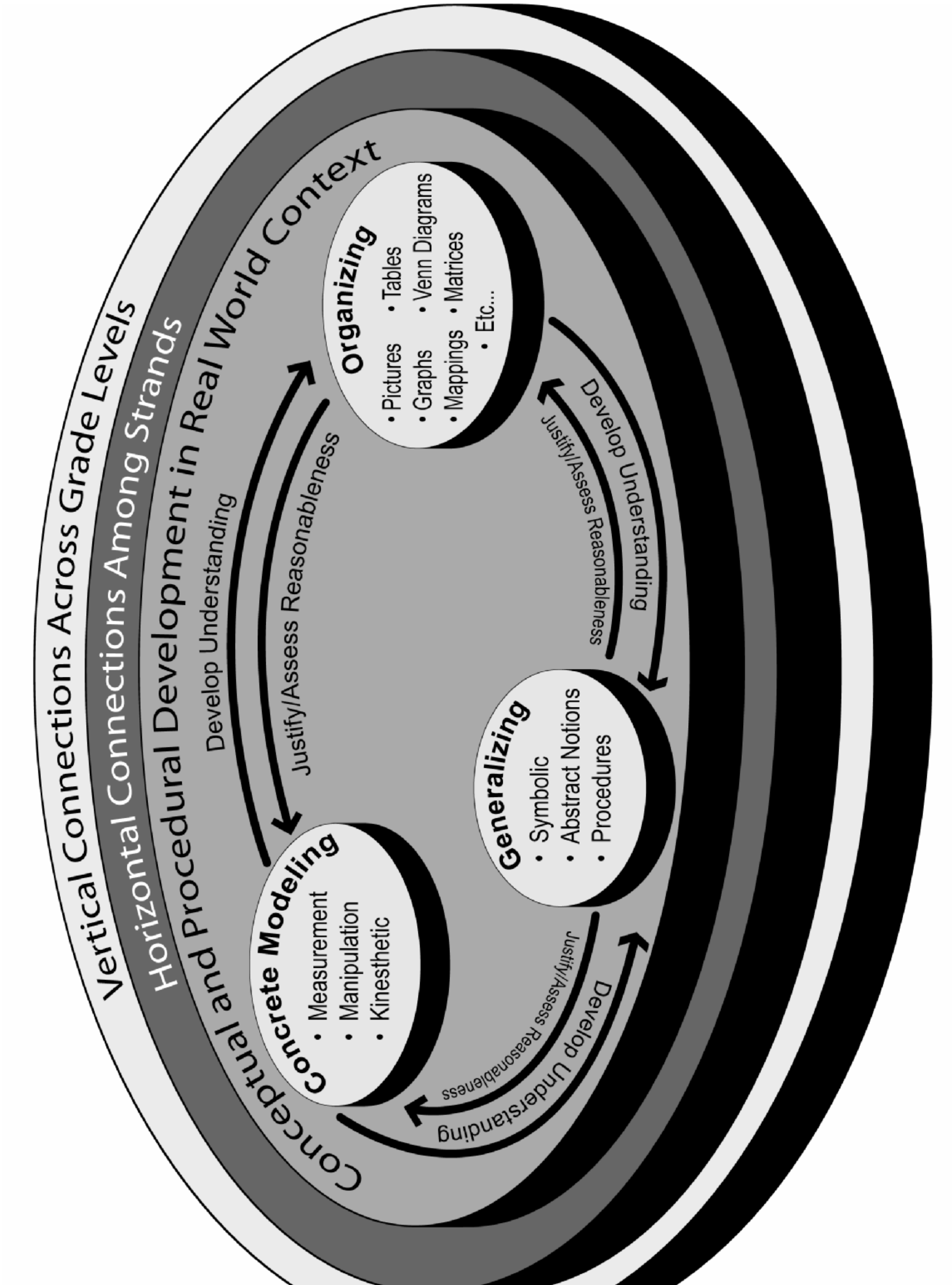
Graph 5



Graph 6



Processing Framework Model



Vocabulary Organizer

My definition	**Personal Association**
Example	**Non-Example**

5E Student Lesson Planning Template

Description	Activity
Engage The activity should be designed to generate student interest in a problem situation and to make connections to prior knowledge. The instructor initiates this stage by asking meaningful questions, posing a problem to be solved, or by showing something intriguing.	
Explore The activity should provide students with an opportunity to become actively involved with the key concepts of the lesson through a guided exploration requiring them to probe, inquire, and question. The instructor actively monitors students as they interact with each other and the activity.	
Explain Students collaboratively begin to sequence events/facts from the investigation and communicate these findings to each other and the instructor. The instructor, acting in a facilitation role, formalizes student findings by providing further explanations and additional meaning or information, such as correct terminology.	
Elaborate Students extend, expand, or apply what they have learned in the first three stages and connect this knowledge with prior learning to deepen understanding. Instructors can use the Elaborate stage to verify students' understandings.	
Evaluate Evaluation occurs throughout students' learning experiences. More formal evaluation can be conducted at this stage. Instructors can determine whether the learner has reached the desired level of understanding the key ideas and concepts.	

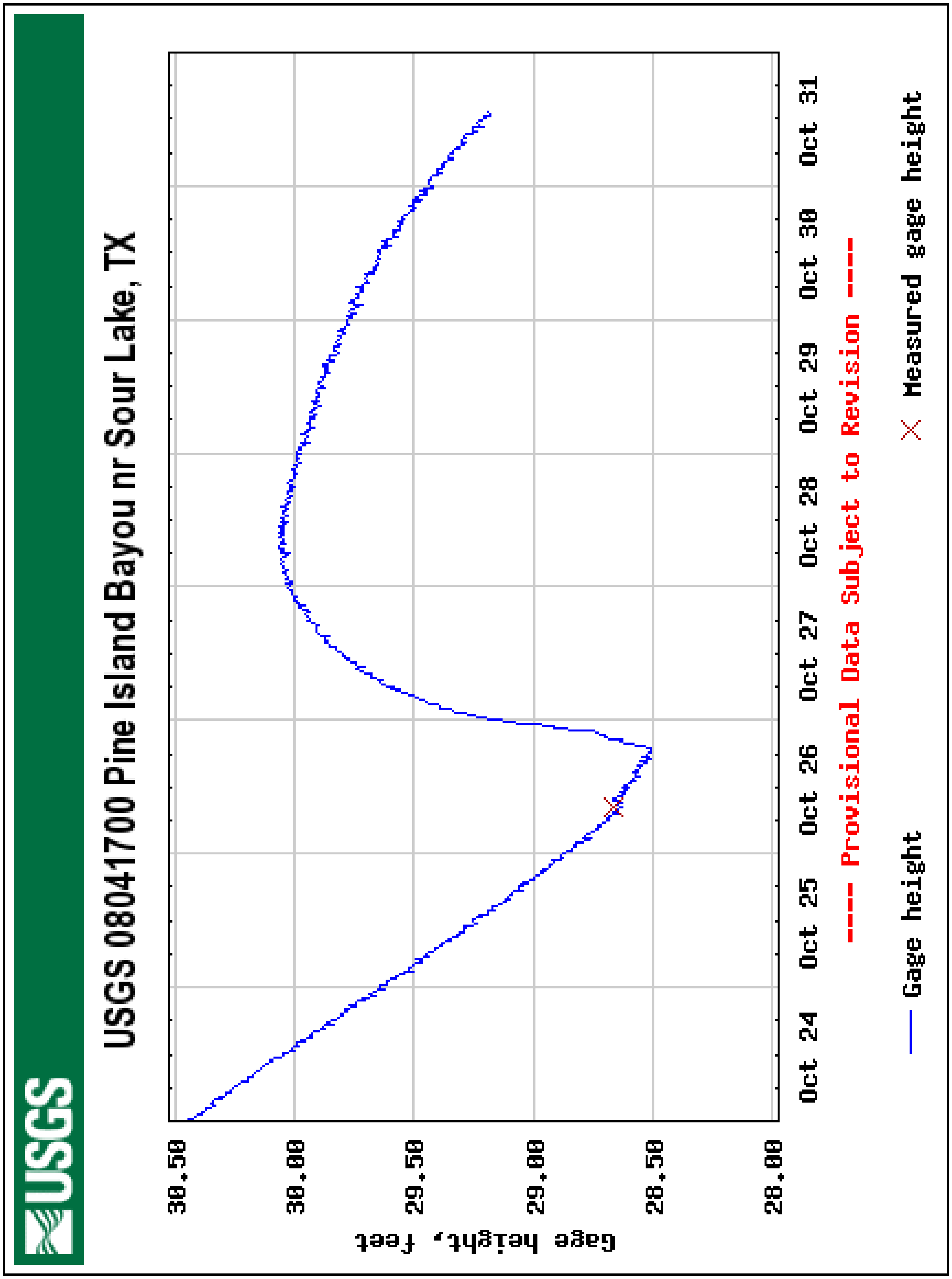
Strategies that Support English Language Learners (ELL)

Strategy	Explore, Explain, Elaborate 1
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond.	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Be conscious of tone and diction. Speak slowly and distinctly.	
Incorporate language skills (reading, writing, speaking, and listening) into instruction.	

Strategies that Support Students with Special Needs

Strategy	Explore, Explain, Elaborate 1
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond.	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Use a system of quick response to needs and accommodations including progress monitoring to inform instruction.	
Accommodate materials for format, structure, sequence, etc. as needed.	

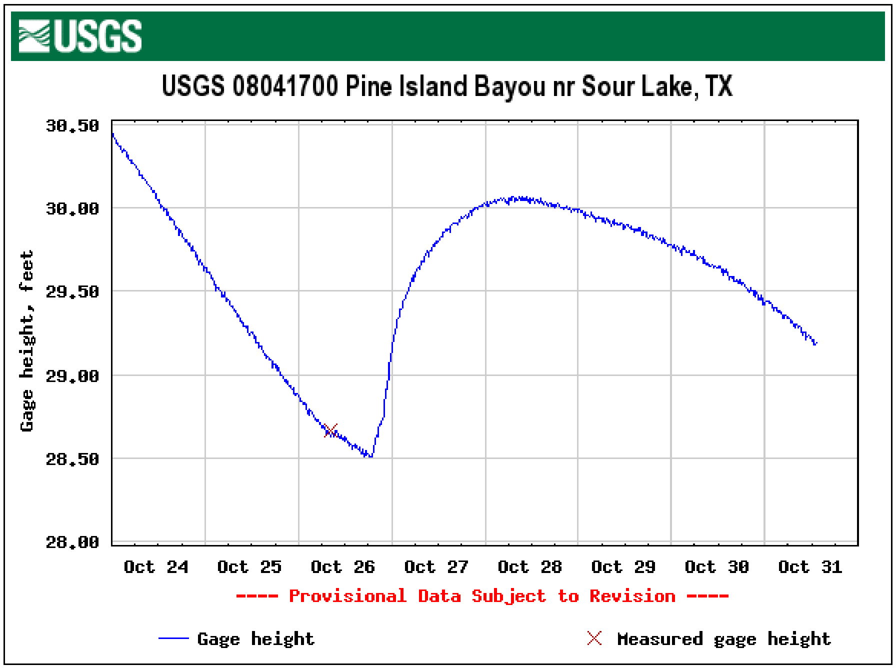
Transparency: Pine Island Bayou



Participant Pages: Inverses of Functions

<u>Part 1</u>: Generating an Inverse Relation

During the Engage phase, you examined graphs of streams' gage heights versus time. Sometimes, hydrologists are concerned about a particular gage height and when the gage for the stream measured that height. In those cases, it makes sense to consider a plot of time versus gage height.



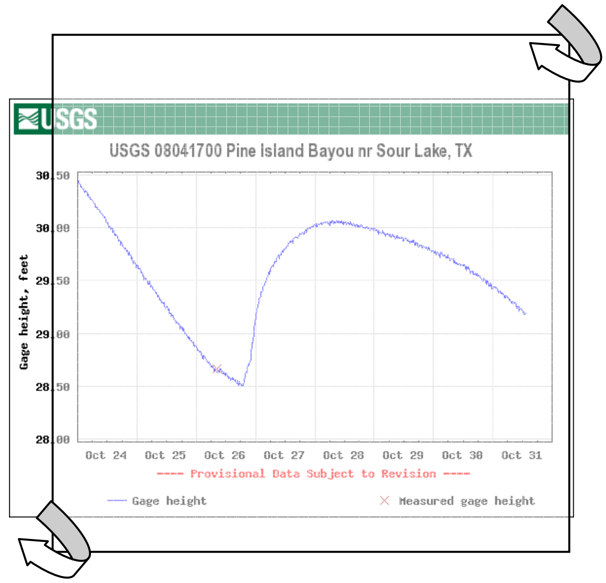
1. For the scatterplot of gage height versus time, what are the inputs and outputs?

2. Does this scatterplot represent a functional relationship? How can you tell?

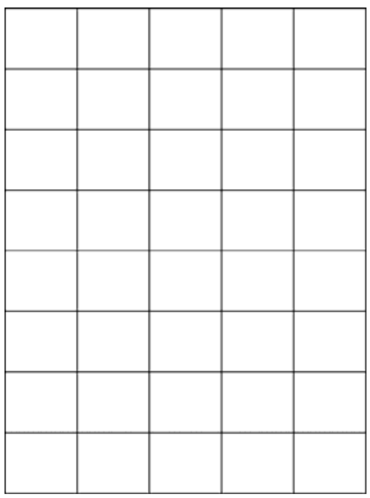
3. What would the graph of the same data look like if we plotted time versus gage height? Sketch a prediction of the graph.

Trace the graph of gage height versus time onto a sheet of patty paper. Be sure to trace the plot and the axes then label your axes on your patty paper.

Take the top-right corner in your right hand and the bottom-left corner in your left hand and flip the patty paper over. Align the origin on the patty paper with the origin of your original graph and align the axes on the patty paper with the axes on the graph.



4. Sketch the resulting graph.



5. How is the new graph similar to the original graph? How are they different?

6. Is the new graph a function? How do you know? Given the situation, does it make sense for the graph to be or not to be a function? Explain your reasoning.

On the activity sheet, **Know When to Fold 'Em**, trace each graph (both curves) onto a piece of patty paper. Be sure to also trace and label the axes.

Fold the patty paper to find all possible lines of symmetry for each pair of graphs shown. Identify the equation for each possible line of symmetry. Record the number of lines of symmetry and their equations for each graph on your recording sheet.

	Total Number of Lines of Symmetry	**Equation(s) of line(s) of symmetry**
Graph 1		
Graph 2		
Graph 3		
Graph 4		
Graph 5		
Graph 6		

7. What patterns do you notice among the lines of symmetry for each of the graphs?

8. Which transformation describes the folds across a line of symmetry for your graphs?

9. Make a summary statement describing the relationship between each pair of curves in the set of given graphs.

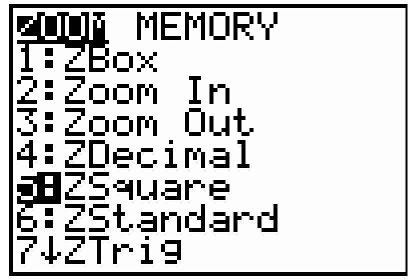
<u>Part 2</u>: Numerical and Symbolic Representations

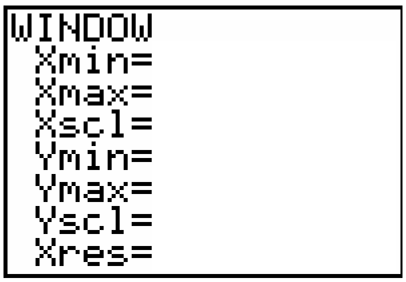
The following coordinates were used to create a geometric design for a quilting pattern.

List 1	1	1	0	0	−1	−1	−2	−1	−2	−1	1
List 2	2	4	5	4	5	4	4	3	3	2	2

1. Create a connected scatterplot of L_2 versus L_1. Sketch your graph and describe your viewing window.

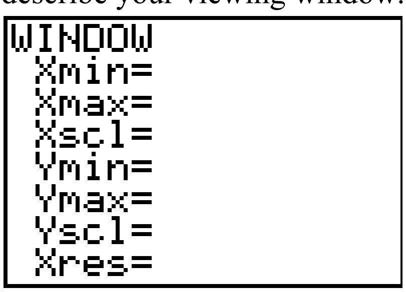
   TIP: Be sure to use a square window. After selecting your appropriate domain and range, use the Zoom-Square feature to square out the grid in your viewing window.





2. What do you think would happen to the graph if we reversed the *x*- and *y*-values?

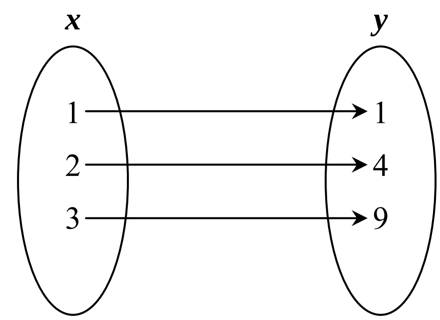
3. Create a second connected scatterplot of L_1 versus L_2. Use a different plot symbol for this scatterplot. Graph this scatterplot with your original scatterplot. Sketch your graph and describe your viewing window. You may need to re-square your viewing window.



4. Compare the two scatterplots. How are they alike? How are they different?

5. How are the x- and y-coordinates related from the first scatterplot to the second scatterplot? How could you represent this relationship symbolically?

Recall that a mapping shows how domain elements for a relation relate, or "map to" their corresponding range elements. For example, the following mapping shows how the x-values $\{1, 2, 3\}$ map to their corresponding y-values $\{1, 4, 9\}$ for the function $y = x^2$.



Enter the functions $Y_1 = 2x - 8$ and $Y_2 = \frac{1}{2}x + 4$ into your graphing calculator.

6. Use the table feature of your graphing calculator to generate values for a mapping for Y1 to show the replacement set for y when $x = \{5, 6, 7, 8, 9\}$.

7. Generate a mapping for Y2 to show the replacement set for y when $x = \{2, 4, 6, 8, 10\}$.

8. How are the two mappings related?

9. In each mapping, to how many y-values does any given x-value map? Would you expect this to be true for other domain and range elements? How do you know?

10. What does this reveal about the relationships in each mapping?

11. In each mapping, how many x-values map to any given y-value? Would you expect this to be true for other domain and range elements? How do you know?

12. What does this reveal about the relationships in each mapping?

13. Examine the graphs of Y1 and Y2. How are they related? (Hint: Be sure you are using a square viewing window.)

Enter the functions $Y_3 = \frac{2}{3}x - 7$ and $Y_4 = \frac{3}{2}x + 7$ into your graphing calculator.

14. Use the table feature of your graphing calculator to generate a mapping for Y3 to show the replacement set for y when $x = \{0, 3, 6, 9\}$.

15. Use the table feature of your graphing calculator to generate a mapping for Y4 to show the replacement set for y when $x = \{-7, -5, -3, -1\}$.

16. How are the two mappings related?

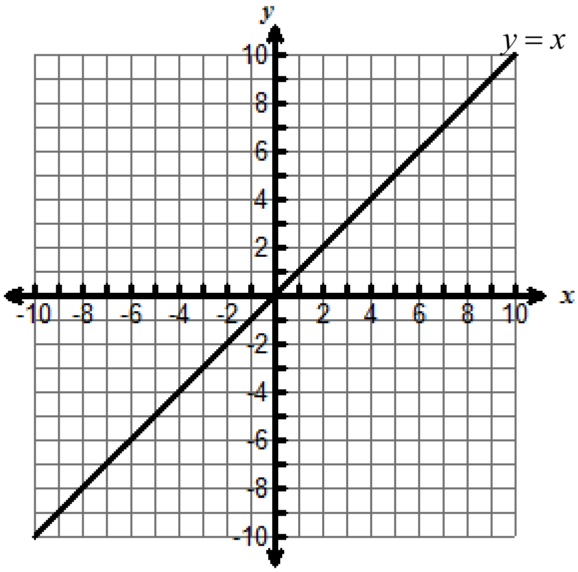
17. Examine the graphs of Y3 and Y4. How are they related? (Hint: Be sure you are using a square viewing window.)

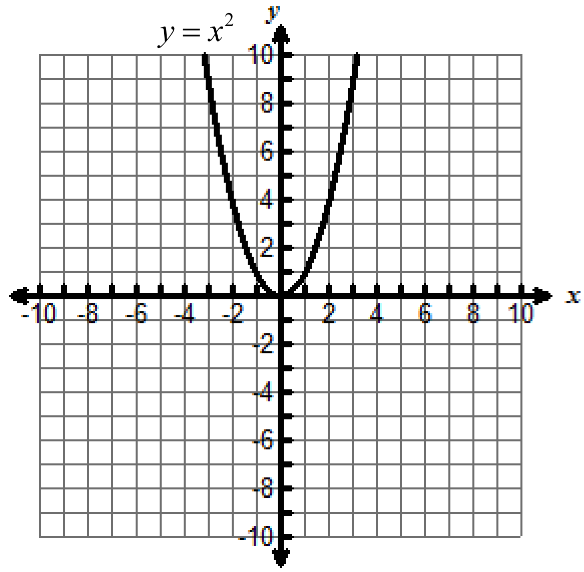
18. The functions in Y1 and Y2 are called "inverse relations" whereas the functions in Y3 and Y4 are not. Based on your mappings, graphs, and equations, why might this be the case?

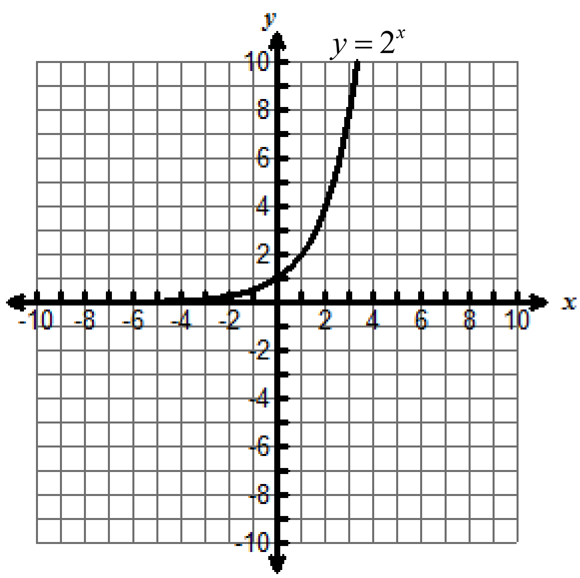
19. Based on your response to the previous question, how might we describe inverse relations graphically, numerically, and symbolically?

<u>Part 3</u>: Investigating Linear Functions

In Algebra 1, students investigate linear, quadratic, and exponential functions. The graphs of the parent functions are shown.

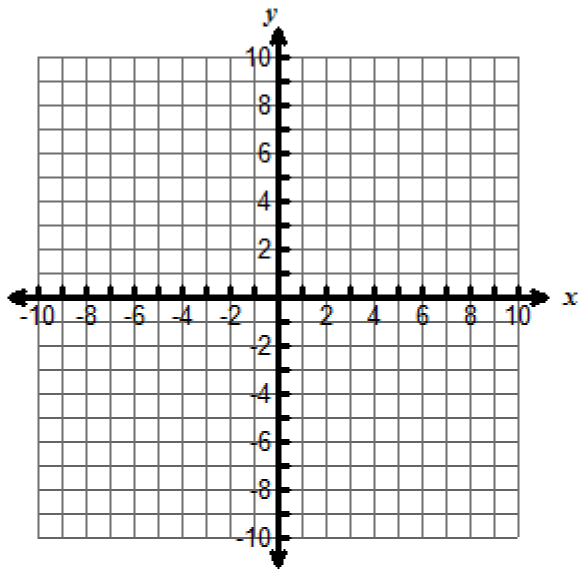






Trace each parent function onto a separate piece of patty paper. Be sure to trace and label the axes as well.

1. Reflect the linear parent function across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of the inverse of the linear parent function? How do they compare with the original function?

3. Is the inverse of the linear parent function also a function? How do you know?

4. What kind of function is the inverse of a linear function?

5. What is the inverse of the function $y = \frac{2}{5}x - 7$? Find the inverse using at least two different methods.

6. How did you determine the inverse?

7. What concepts and procedures did you apply to determine the inverse?

8. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{2}{5}x - 7$?

9. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{2}{5}x - 7$?

10. How do these two sets of operations compare?

11. Describe the graph of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent functions.

12. Describe the graph of the inverse of the function $y = \frac{2}{5}x - 7$ in terms of transformations of the parent functions.

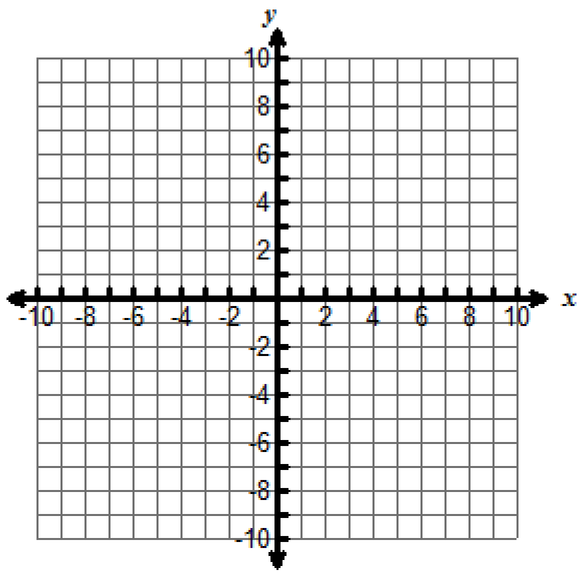
13. Compare the graphs of $y = \frac{2}{5}x - 7$, its inverse, and the line $y = x$.
    a. What do you notice about the intercepts of the graphs?

    b. Where are the three graphs concurrent? What is the significance of this point?

    c. In terms of transformations, how do the graphs of the original function and its inverse compare?

<u>Part 4</u>: Quadratic Functions

1. Reflect the quadratic parent function across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of the inverse of the quadratic parent function? How do they compare with the original function?

3. Is the inverse of the quadratic parent function also a function? How do you know?

4. If the inverse is not a function, how can we restrict the domain and/or range of the original function so that the inverse is also a function?

5. What kind of function is the inverse (range restricted)of a quadratic function?

6. Is either the original parent function or its range restricted inverse one-to-one? How do you know?

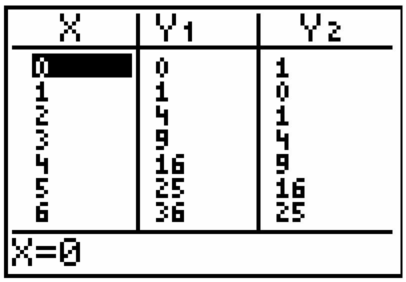
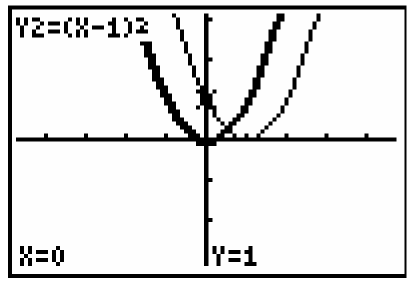
7. What is the inverse function of the function $y = -3(x-1)^2 + 2$? Find the inverse using at least two different methods.

8. How did you determine the inverse?

9. What concepts and procedures did you apply to determine the inverse?

10. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = -3(x-1)^2 + 2$?

11. From the perspectives of number operations and transformations, develop the quadratic function, $y = -3(x-1)^2 + 2$, from the parent function $y = x^2$. Include tabular, graphical, and symbolic representations of the number operation as it applies to the function.

<table>
<tr><th>Number Operation</th><th>Tabular</th><th>Graphical</th><th>Symbolic</th></tr>
<tr><td>Subtract 1 from the x-value in the parent function $y = x^2$.</td><td>
</td><td>
</td><td>$y = (x-1)^2$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
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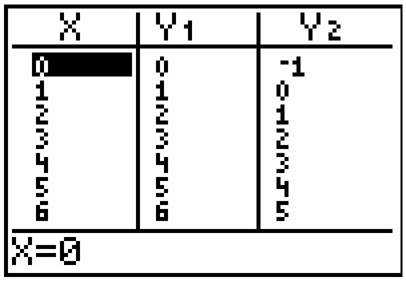
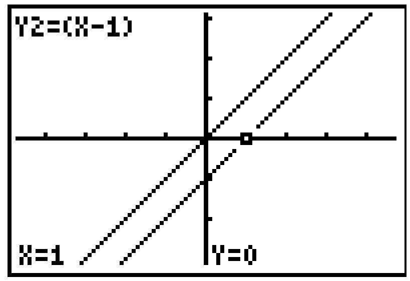
</table>

12. Now, use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the quadratic function, $y = -3(x-1)^2 + 2$, from $y = x$.

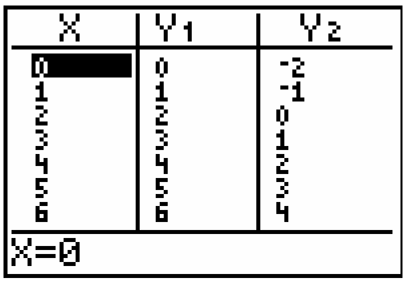
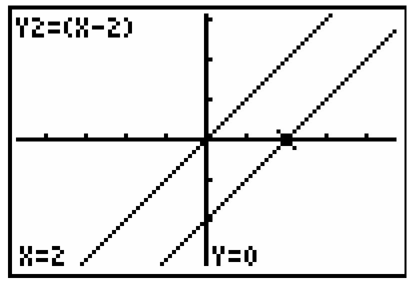
<table>
<tr><th>Number Operation</th><th>Tabular</th><th>Graphical</th><th>Symbolic</th></tr>
<tr><td>Subtract 1 from the parent function $y = x$</td><td>
</td><td>
</td><td>$y = x - 1$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
</table>

13. How does each successive number operation transform the function numerically, graphically, and symbolically?

14. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = -3(x-1)^2 + 2$?

15. How does this set of operations compare to the operations applied to generate the function $y = -3(x-1)^2 + 2$?

16. Use number operations to describe what happens to the domain elements, represented by the variable x, as you develop the square root inverse of the function, $y=-3(x-1)^2+2$, from $y=x$.

Number Operation	Tabular	Graphical	Symbolic
Subtract 2 from the parent function *y* = *x*.			$y=x-2$

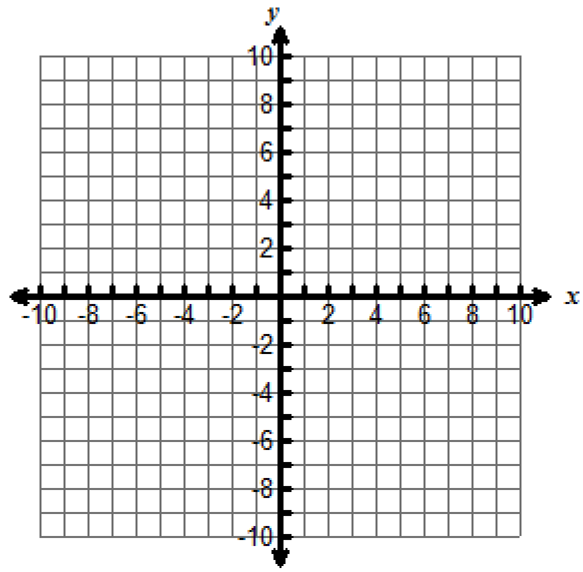
17. How do the inverse operations relate to the inverse function?

18. Generalize how inverse relations for quadratic functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.

Number Operation	**Tabular**	**Graphical**	**Symbolic**

<u>Part 5:</u> Exponential Functions

1. Reflect the exponential parent function, $y = 2^x$, across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of $y = 2^x$?

3. What is the domain and range of the inverse of $y = 2^x$? How do they compare with the original function?

4. What asymptote(s) does the original function, $y = 2^x$ have? Why does this asymptote exist?

5. What asymptote(s) does the inverse of $y = 2^x$ have? How do they compare to the asymptotes of the original function?

6. Is the inverse of $y = 2^x$ also a function? How do you know?

7. What kind of function is the inverse of an exponential function?

8. Is either the original parent function or its inverse one-to-one? How do you know?

9. What is the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$? Find the inverse using at least two different methods.

10. How did you determine the inverse?

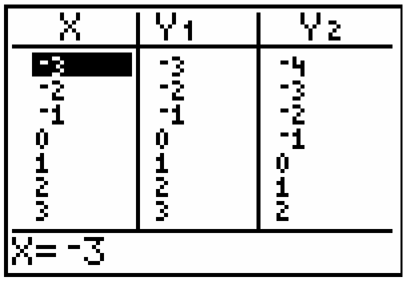
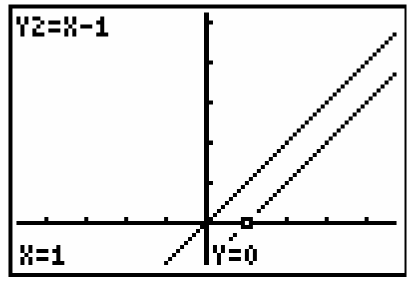
11. What concepts and procedures did you apply to determine the inverse?

12. Numerically, what operations are being done to the domain values to generate the corresponding range values in the function $y = \frac{1}{2}(10)^{x-1} + 3$?

13. Numerically, what operations are being done to the domain values to generate the corresponding range values in the inverse of the function $y = \frac{1}{2}(10)^{x-1} + 3$?

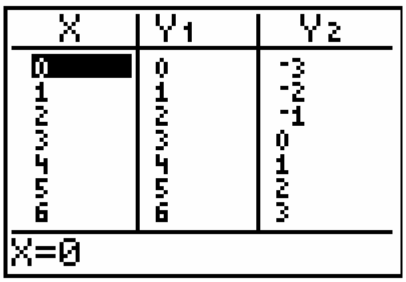
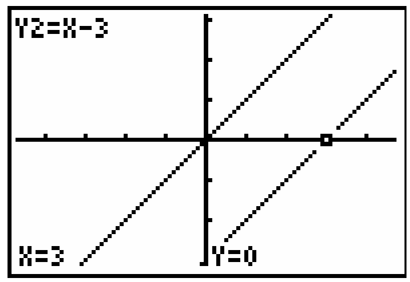
14. How do these two sets of operations compare?

15. Use number operations to describe what happens to the domain elements, represented by the variable x, as you develop the exponential function, $y = \frac{1}{2}(10)^{x-1} + 3$, from $y = x$.

Number Operation	**Tabular**	**Graphical**	**Symbolic**
Subtract 1 from the parent function $y = x$			$y = x - 1$

16. How does each successive number operation transform the function numerically, graphically, and symbolically?

17. Use number operations to describe what happens to the domain elements, represented by the variable *x*, as you develop the logarithm inverse of the function, $y = \frac{1}{2}(10)^{x-1} + 3$, from $y = x$.

Number Operation	**Tabular**	**Graphical**	**Symbolic**
Subtract 3 from the parent function $y = x$			$y = x - 3$

18. How do the inverse operations relate to the inverse function?

19. Generalize how inverse relations for exponential functions compare to their corresponding original functions. Consider each of the four representations and use the table below to record your responses.

Number Operation	Tabular	Graphical	Symbolic

<u>Part 6</u>: Compositions of Functions

Enter the function $y = 2x - 1$ into *Y1* of your graphing calculator's function editor. Enter the inverse of this function into *Y2*. Enter the composition of *Y2* and *Y1* into *Y3*.

1. Sketch the graphs of the three functions (describe your viewing window). What relationships and patterns do you notice?

```
WINDOW
 Xmin=
 Xmax=
 Xscl=
 Ymin=
 Ymax=
 Yscl=
 Xres=
```

2. Look at the table values for each of the three functions. What do you notice?

3. How could you represent the composition of these two functions with a mapping?

4. What effect does composition of inverse functions have? Why do you think this is so?

5. Does the order of composition matter? Explain your answer.

6. Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse function. Describe the result numerically, graphically, and symbolically.

<u>Part 7</u>: Extension

1. Investigate the composition of a quadratic function such as $y = x^2 - 2$ and its inverse <u>relation</u>. Describe the result numerically, graphically, and symbolically.
   Case 1: Inverse is not a function – Composition of Inverse(Original)

   Case 2: Inverse is not a function – Composition of Original(Inverse)

2. Based on your experiences with linear and quadratic functions, what would you expect to be true about compositions of other types of functions, such as exponential, rational, or polynomial functions? Give examples or counterexamples.

Leader Notes: Watching the Beat Go On

Purpose:
The purpose of this section of the professional development is to investigate stumbling blocks and challenges that students and teachers have in developing concepts and procedures involved with learning about square root functions.

Descriptor:
The Explore phase has seven parts. The first part sets the stage by introducing the metronome as an instrument to keep musical time. In the second part participants will collect data during a simple simulation of a metronome using a "weight" and "spring" (plastic bottle of water and rubber bands) by measuring the period of the "ticks." In part 3 participants will analyze the data to establish the inverse relationship between square root and quadratic functions. Then participants will look at transformations to the square root function $f(x)=\sqrt{x}$ using transformation form ($y=a\sqrt{(x-h)}+k$). Part 5 is an extension in which participants may investigate transformations as a result of changes in the coefficient of *x*. In part 6 participants will connect the changing roles of *a*, *h,* and *k* in square root and quadratic functions. Finally, participants will explore several methods for solving square root equations and inequalities in literal and contextual situations.

Duration:
2 hours

TEKS:
a5 Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.9 **Quadratic and square root functions.** The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

2A.9A The student is expected to use the parent function to investigate, describe, and predict the effects of parameter changes on the graphs of square root functions and describe limitations on the domains and ranges.

2A.9B The student is expected to relate representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

2A.9C The student is expected to determine the reasonable domain and range values of square root functions, as well as interpret and determine the reasonableness of solutions to square root equations and inequalities.

2A.9D The student is expected to determine solutions of square root equations using graphs, tables, and algebraic methods.

2A.9E The student is expected to determine solutions of square root inequalities using graphs and tables.

2A.9F The student is expected to analyze situations modeled by square root functions, formulate equations or inequalities, select a method, and solve problems.

2A.9G The student is expected to connect inverses of square root functions with quadratic functions.

TAKS™ Objectives Supported:

While the Algebra II TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 5: Quadratic and Other Nonlinear Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance: Copies of participant pages

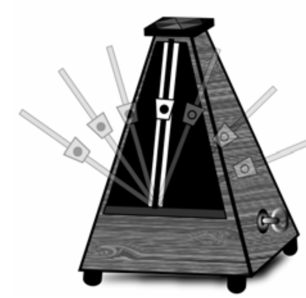
Presenter Materials: Overhead graphing calculator, metronome or metronome video

Per group: Filled plastic 500mL bottle, rubber bands (enough to loop together to make a “spring” about 1 meter long), metric tape measure, stopwatch, cards for **Transformations of Square Root Functions Card Sort (**cut apart and placed in plastic bags)

Per participant: Copy of participant pages, graphing calculator

Explore

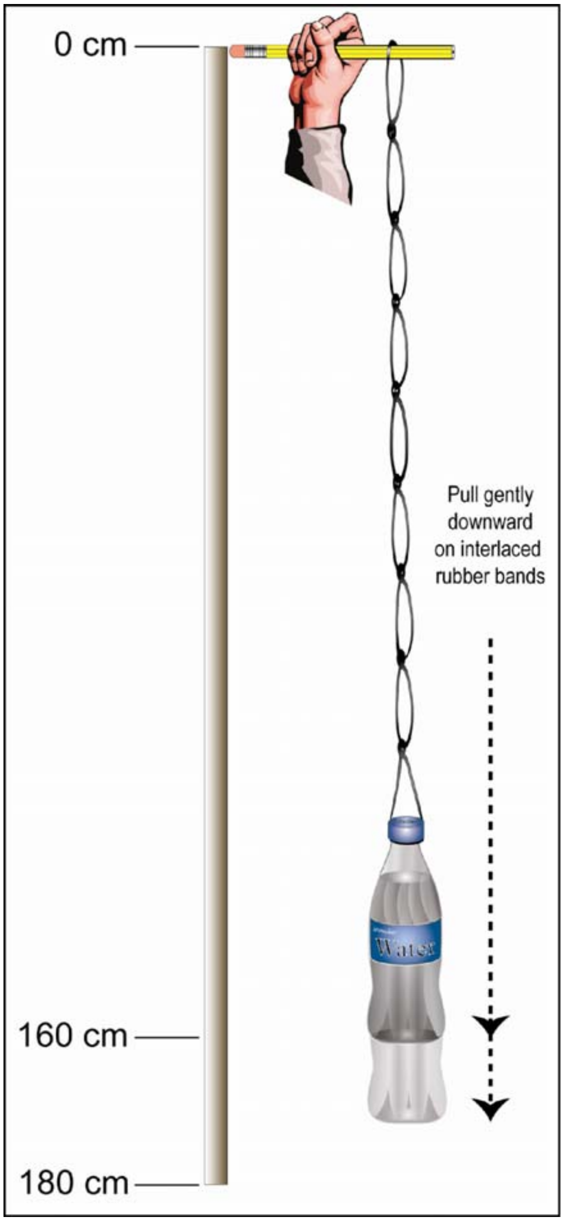
<u>Part 1</u>: Setting the Stage



Leader Notes: If you have access to a metronome, you might want to demonstrate how it works. If you do not have a metronome, a video clip of a pianist playing a piece of music at several speeds with a metronome is included on the presenter CD.

A metronome frequently is used in music to mark exact time using a repeated tick. The frequency of the ticks varies in musical terms from slow *(largo,* about 40-60 beats per minute) to fast *(presto,* 168-208 beats per minute). Individual instrumentalists, choirs, bands, and orchestras all use metronomes to ensure that the beat of the music is consistent with the instructions of the composer and does not unintentionally speed up or slow down while the piece is being played.

What do you notice about the frequency of the ticks and setting of the weight as the music is being played?
The closer the weight is to the base, the faster the ticks and the faster the beat of the music. Conversely, the farther the weight is from the base, the less frequent the intervals and the slower the beats of the music.



Part 2: Modeling and Gathering Data

In this phase of the professional development participants investigate the square root function generated by modeling the period of 10 metronome ticks. In this simplified experiment, use a filled 500 mL bottle suspended from a pencil by a 1-meter strand of rubber bands. (This may vary depending on the elasticity of the rubber bands. When the bottle is attached, the stretched rubber band spring should extend so the bottom of the bottle of water is at least 160 cm below the pencil.) The pencil is held at a height of 160 to 200 centimeters above the floor by a participant, eraser end of the pencil held firmly against the wall so the pencil will not move. The rubber band spring should be attached close to the opposite end of the pencil so the bottle can bounce without striking the wall. To simulate moving the weight up or down on the metronome, the rubber band spring will be wrapped around the pencil until the bottom of the bottle is raised to the desired height. All participants should use the same size bottle, filled to the same height, and the same size rubber bands so the data will be as consistent as possible.

In this investigation, a metric tape measure should be taped to the wall. Participants will suspend a filled 500 mL bottle by a 1-meter rubber band spring from a pencil firmly held perpendicular to a wall at a height of 180-200 centimeters. The difference between the length of the rubber band spring and the suspension height should allow the bottle to bounce without touching the floor.

The length of the rubber band spring will be shortened by wrapping the spring around the pencil so that data is collected at approximately 20-centimeter intervals.

Divide participants into groups of 4. Each person in the group has a job.

Materials manager: Gets the necessary materials, directs the team in setting up the investigation, holds the pencil with the suspended bottle, and shortens the spring when needed.

Measures manager: Measures the distance from the pencil to the bottom of the bottle for each length, initiates the bounce by pulling the suspended bottle down an additional 10 centimeters and counts the bounces (10 at each height).

Time manager: Uses a stop watch to determine the length of each 10-bounce period of time. The time starts when the bottle is released by the measures manager and ends when the bottle completes its 10^{th} bounce.

Data manager: Records the necessary measurements in the table and shares the data with the team.

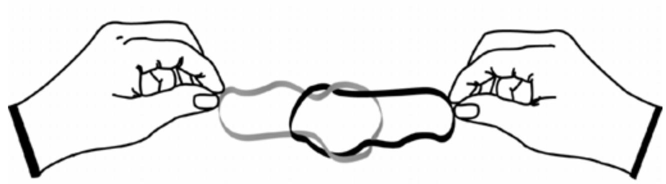
Set-up Instructions

Step 1. **The materials manager should get the necessary materials and ask 2 of the team members to secure the tape measure against the wall. The tape measure should be positioned perpendicular to the floor so that the "zero end" is at 180-200 centimeters above the floor.**

Step 2. **While the tape measure is being positioned, the materials manager and remaining team member(s) build the rubber band spring by looping rubber bands together until the length of the spring is about 1 meter. This task will go more quickly if each person makes part of the spring. Then the pieces can be joined.**

Stretch bands between thumb, forefinger and middle finger.

Merge bands by grasping left band with right forefinger pulling it toward the center.



Pull both bands outward to form an intertwined loop.

Step 3. **Secure one end of the rubber band spring to the pencil and the other around the neck of the bottle. It also works to remove the cap, insert the end of the spring in the bottle, and screw the cap back on.**

Step 4. **The measures manager secures the spring so that it is approximately 160 centimeters in length. After the bottle remains motionless for a few seconds, he should measure the actual length of the spring. The length of the spring includes the length of the rubber band and the length of the bottle.**

Step 5. **The measures manager then pulls the bottle downward about 10 cm and releases it.**

Step 6. **The time manager starts the stopwatch when it is released and stops it at the end of 10 complete bounces. It may be helpful to have all team members count aloud together.**

Step 7. **The data manager records the number of seconds in the table under Trial 1 for 160 cm.** ***Hint: It is more meaningful to start with the spring fully extended and to shorten the spring than to begin at the top and work down.***

Step 8. **Repeat for Trials 2 and 3. Average the data from the 3 trials and record in the Average Time column.**

Step 9. **The materials manager who is holding the pencil shortens the spring by wrapping it around the pencil until the desired length of 140 is obtained.**

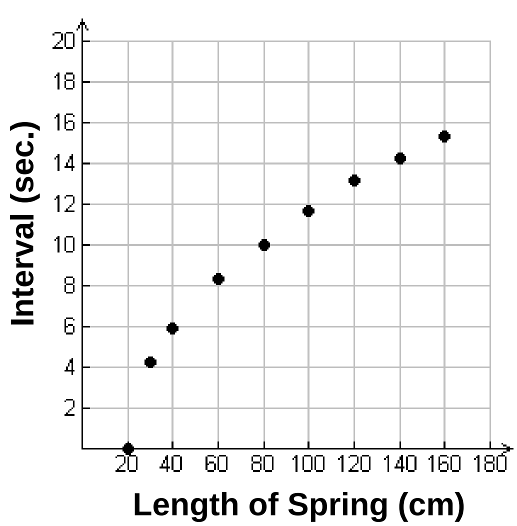
Step 10. **Continue repeating the procedure with shortened lengths of rubber band spring. Continue to record your data.**

1. Record your data in the table below.

Sample data using a bottle 20 cm tall:

Approximate Length of Spring (cm) *x*	Actual Length of Spring (cm) *x*	Trial 1	Trial 2	Trial 3	Average Time (sec) *y*
0	*0*				*Cannot be done*
20	*20*				*0*
30	*30*				*4.23*
40	*40*				*5.92*
60	*60*				*8.30*
80	*80*				*10.04*
100	*100*				*11.70*
120	*120*				*13.17*
140	*140*				*14.22*
160	*160*				*15.34*

2. Make a scatterplot of the data you collected.



3. **What is the independent variable?**
   the length of the spring in centimeters

4. **What is the dependent variable?**
   the period of 10 bounces measured in seconds

5. **Write a dependency statement relating the two variables.**
   The interval of a bounce depends on the length of the rubber band spring.

6. **What is a reasonable domain for the set of data?**
   Responses may vary. The length of the extended rubber band spring with the bottle is approximately 160 cm.

7. **What is a reasonable range for the set of data?**
   Responses may vary. The size and elasticity of the rubber bands used will affect the data. The maximum 10-bounce period appears to be about 16 seconds for this set of data.

8. **Is this data set continuous or discrete? Why?**
   The data set is discontinuous; however, the data points would be closer if we were to shorten the spring by smaller increments. The data theoretically could be continuous, but it is discontinuous in any practical collection of data.

9. **Does the set of data represent a function? Why?**
   Yes, the data set represents a function. For each increase in the length of the spring, there is an increase in the 10-bounce interval.

10. **Write a summary statement about what happened in this data investigation.**
    The shorter the spring, the shorter the interval appears to be. The interval time increases rapidly between intervals when the spring is at short lengths but increases more slowly as the length of the spring increases.

11. **Is the function increasing or decreasing?**
    Increasing

12. **Is the rate of change constant?**
    No, the function values increase rapidly at first and then more slowly.

13. **Does the data you collected appear to be a linear, quadratic, exponential, or some other type of parent function? Why?**
    The data does not appear to be linear because it is not constant. It does not appear to be quadratic because of its behavior in relation to the y-axis, and it seems to lack a symmetric "half." It does not appear to be exponential because its rate of change slows instead of continuing to increase by a constant factor. Based on our explorations in Explore/Explain/Elaborate 1, it might be a square root function because it appears to be the inverse of a quadratic function.

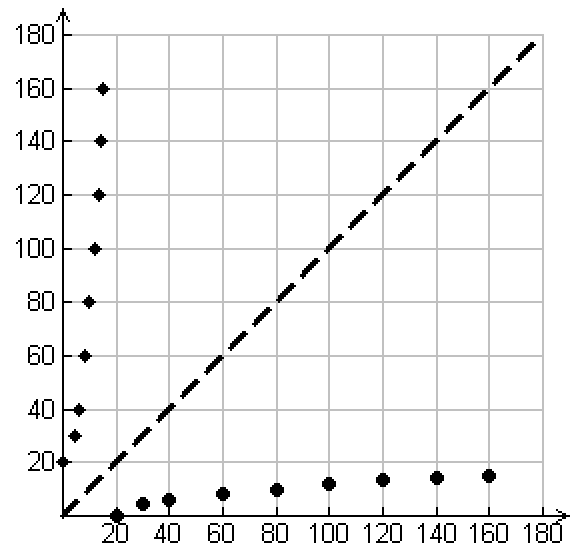
14. How is the bottle bounce activity similar to the ticks of a metronome?
The intervals of the bottle bounce become longer as the spring becomes longer. The intervals of the ticks of the metronome become longer as the length of the rod below the weight becomes longer.

15. What kind of function do you think models the ticking of a metronome? Why?
Probably a square root function because the shorter the rod, the more frequent the tick of the metronomes, just as the shorter the rubber band spring, the more frequent the bounces of the bottle.

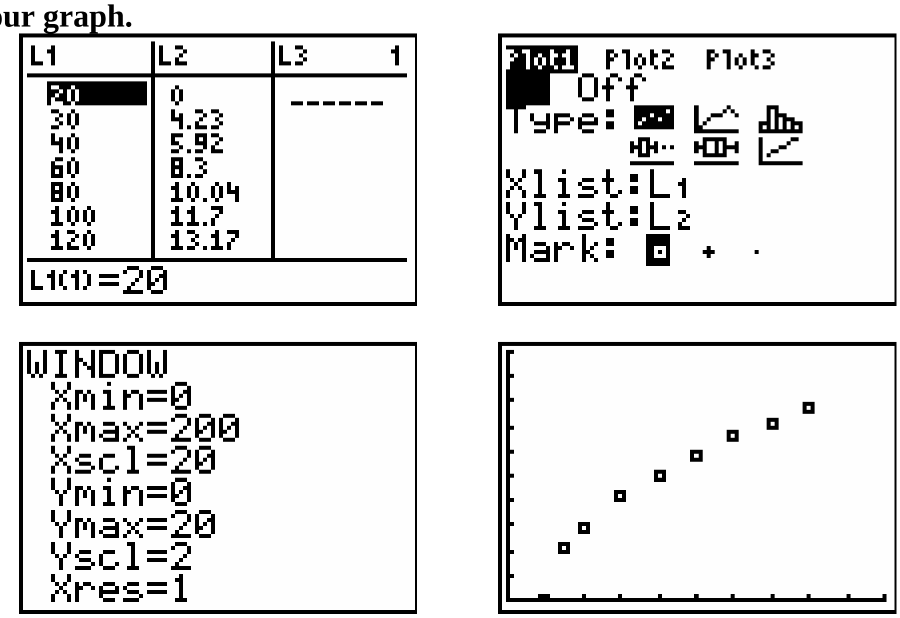
<u>Part 3</u>: Analyzing the Data

1. **How could you determine whether this function is the inverse of another parent function?**
   Answers may vary. The data could be reflected over the line y = x. That is, the x values would become the y values, and vice versa. Or, the L_2 values could be mapped to L_1 using the table feature of the graphing calculator.

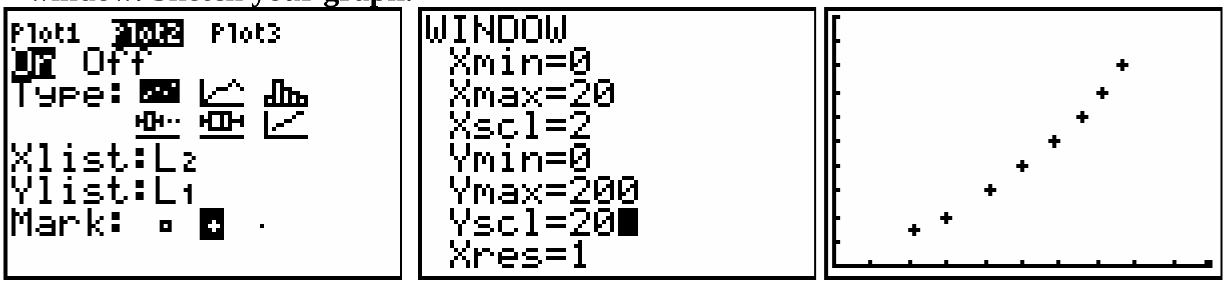
Graphing data from Part 2 and inverse of data simultaneously.



2. **Input your values into L_1 and L_2 of a graphing calculator, letting L_1 be independent values and L_2 dependent values, and create a scatterplot of the original data. Sketch your graph.**



3. **Create a second scatterplot that represents an inverse of the data. Use a different plot symbol for this scatterplot. Determine a new domain and range, and set a new viewing window. Sketch your graph.**



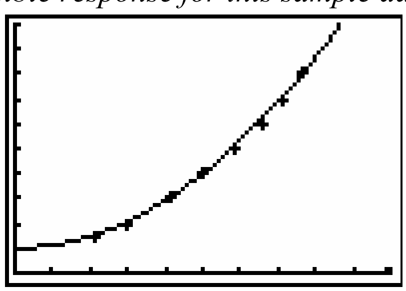
4. **What changes must you make to the window to view the second set of data?**
   reverse the x and y values

5. **Which parent function do the *reflected* points most closely appear to represent?**
   quadratic

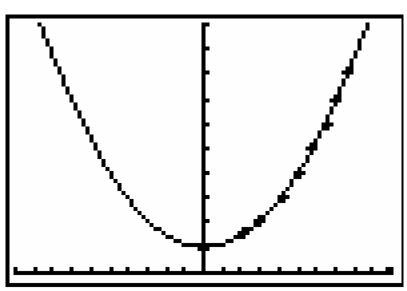
6. **How did you determine your function?**
   Answers may vary.

7. **How might you confirm your conjecture?**
   Answers may vary. Sample response: By checking values in the table.

8. **Without using regression, find a function that approximates your data for Plot 2.**
   Answers may vary. A possible response for this sample data is $y = 0.6x^2 + 20$.



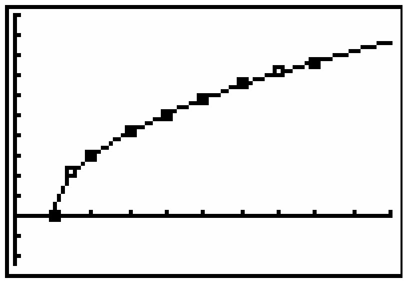
9. **Does your viewing window allow you to see both sides of the parabola? If not, readjust your viewing window.**
   Possible viewing window:



10. How could you use this function to find a function that would approximate the first scatterplot you graphed?

I could find the inverse of the function; in this case, $y = \sqrt{\dfrac{(x-20)}{0.6}}$.

11. Reset your window to view Plot 1. Enter the equation you found in question 8 in the equation editor. Is your graph a close fit to the data in Plot 1?
Possible viewing window representing this sample data



12. Compare and contrast the graphs of a quadratic function and a square root function. How are the graphs similar, and how are they different?
The square root function and the positive side of the quadratic function are inverses. There is no "negative" side to the square root function.

13. Why are there no negative coordinates in the square root function?
Possible responses: If the "half" of the quadratic function were graphed, it would not be a function. It would fail the vertical line test. For this real-life situation, we can have only real numbers as our values.

14. What is the domain of this square root function?
$x \geq 0$

15. What is the range of this square root function?
$y \geq 0$

16. What conclusions can you make about the attributes of a square root function?
Answers may vary but should indicate an understanding of the characteristics of the square root parent function, such as limitations of domain and range.

17. What conclusions can you make about the collected data?
Answers may vary. It can be represented by a square root function.

<u>Part 4</u>: Making Symbolic Generalizations

Distribute Activity Sheet A to each participant and one baggie of equations. Pairs will work together to sort the equations into the proper row and column.

Transformations of Square Root Functions Card Sort

1. Place the cards in the proper row and column.

Description	Example	Example	Notation
Vertical Translation Up	$y=\sqrt{x}+5$	$y=\sqrt{x}+1$	$y=\sqrt{x}+k$ $k>0$
Vertical Translation Down	$y=\sqrt{x}+(-3)$	$y=\sqrt{x}-5$	$y=\sqrt{x}+k$ $k<0$
Horizontal Translation Left	$y=\sqrt{x-(-4)}$	$y=\sqrt{x+5}$	$y=\sqrt{x-h}$ $h<0$
Horizontal Translation Right	$y=\sqrt{x-(+6)}$	$y=\sqrt{x-5}$	$y=\sqrt{x-h}$ $h>0$
Vertical Stretch	$y=3\sqrt{x}$	$y=5\sqrt{x}$	$y=a\sqrt{x}$ $a>1$
Vertical Compression	$y=\frac{2}{3}\sqrt{x}$	$y=\frac{1}{5}\sqrt{x}$	$y=a\sqrt{x}$ $0<a<1$
Reflection	$y=-\sqrt{x}$	$y=-5\sqrt{x}$	$y=a\sqrt{x}$ $a<0$

2. **Describe the role of *a*.**
   $|a|$ is the vertical stretch or compression factor. If a<0, the graph is reflected across the x-axis.

3. **Describe the role of *h*.**
   h is the horizontal translation.

4. **Describe the role of *k*.**
   k is the vertical translation.

5. **Using *x, a, h* , and *k*, write an equation that could be used to summarize the transformations to the square root function.**
   $y = a\sqrt{x-h}+k$ *or, using function notation*, $f(x) = a\sqrt{x-h}+k$

6. **Revisiting the bottle bounce investigation, describe the transformation to the square root parent function that represents your data.**
   Answers may vary.

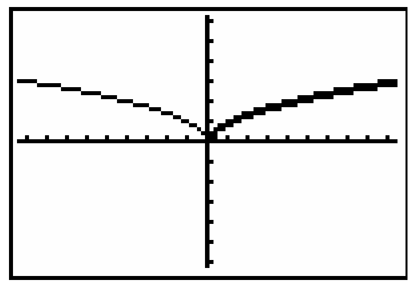
<u>Part 5 (Optional Extension)</u>: Investigating the Coefficient of *x*

Leader notes: Horizontal stretches, compressions, and reflections are explored in Precalculus. However, some participants may respond that the transformation form is $f(x) = a\sqrt{\frac{1}{b}(x-h)} + k$ with $|b|$ being the horizontal stretch or compression factor, and that if $b < 0$, the graph is reflected across the y-axis.

1. Using $f(x) = a\sqrt{\frac{1}{b}(x-h)} + k$, predict the changes in the parent function for the following functions. Then check with your graphing calculator.

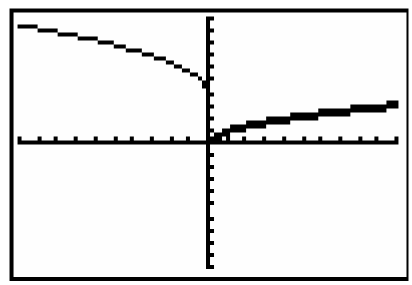
a. $f(g) = \sqrt{-x}$

The graph of the parent function will be reflected across the y-axis.



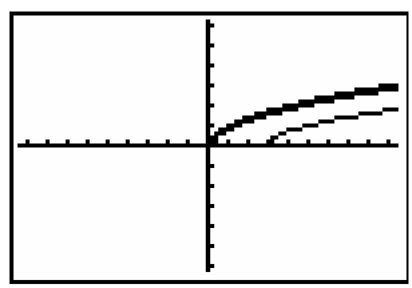
b. $f(g) = \sqrt{-3x} + 4$

The graph of the parent function will be reflected across the y-axis, horizontally compressed by a factor of $\frac{1}{3}$*, and then translated 4 units up.*



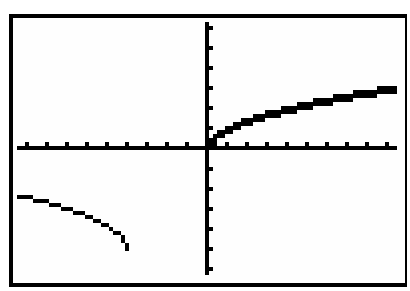
c. $f(g) = \sqrt{\frac{1}{2}(x-3)}$

The graph of the parent function will be horizontally stretched by a factor of 2 and translated 3 units to the right.



d. $f(g) = 2\sqrt{-\frac{1}{3}(x+4)} - 5$

The graph of the parent function will be vertically stretched by a factor of 2, reflected across the y-axis, horizontally stretched by a factor of 3, translated 4 units left, and translated 5 units down. (Whew!)



2. What can you summarize about transformations of the square root parent function as a result of changes to $\frac{1}{b}$?

$|b|$ is the horizontal stretch or compression factor. If $b<0$, the graph is reflected across the y-axis.

<u>Part 6 (Optional Extension)</u>: Connecting the Roles of *a, h,* and *k* in Square Root and Quadratic Functions

Display Transparency A. Ask participants to find the inverse of each equation algebraically.

Equation 1 $y = 3\sqrt{x-5} + 6$

$$x = 3\sqrt{y-5} + 6$$
$$x - 6 = 3\sqrt{y-5} + 6 - 6$$
$$x - 6 = 3\sqrt{y-5}$$
$$\frac{x-6}{3} = \frac{3\sqrt{y-5}}{3}$$
$$\frac{x-6}{3} = \sqrt{y-5}$$
$$\left(\frac{x-6}{3}\right)^2 = \left(\sqrt{y-5}\right)^2$$
$$\left(\frac{x-6}{3}\right)^2 = y - 5$$
$$\left(\frac{x-6}{3}\right)^2 + 5 = y - 5 + 5$$
$$\left(\frac{x-6}{3}\right)^2 + 5 = y$$

or

$$y = \left(\frac{1}{3}\right)^2 (x-6)^2 + 5$$

Equation 2 $y = 7(x-2)^2 + 4$

$$x = 7(y-2)^2 + 4$$
$$x - 4 = 7(y-2)^2 + 4 - 4$$
$$x - 4 = 7(y-2)^2$$
$$\frac{x-4}{7} = \frac{(y-2)^2}{7}$$
$$\frac{x-4}{7} = (y-2)^2$$
$$\pm\sqrt{\frac{x-4}{7}} = \sqrt{(y-2)^2}$$
$$\pm\sqrt{\frac{x-4}{7}} = y-2$$
$$\pm\sqrt{\frac{x-4}{7}} + 2 = y-2+2$$
$$\pm\sqrt{\frac{x-4}{7}} + 2 = y$$

or

$$y = \frac{1}{\sqrt{7}}\left(\sqrt{x-4}\right) + 2 \quad \text{or} \quad y = -\frac{1}{\sqrt{7}}\left(\sqrt{x-4}\right) + 2$$

1. **Find the inverse of Equation 1.**

2. **Numerically and graphically compare and contrast Equation 1 and its inverse.**
   Answers will vary but should demonstrate an understanding that the inverse of the square root equation in Example 1 is a quadratic function that is translated 6 units right; the square root function is shifted 6 units up. The quadratic function is shifted 5 units up; the square root function is shifted 5 units to the right. The quadratic equation is vertically compressed by a factor of $\left(\frac{1}{3}\right)^2$ *while the square root equation is vertically stretched by a factor of 3.*

3. **Find the inverse of Equation 2.**

4. **Numerically and graphically compare and contrast Equation 2 and its inverse.**
   Answers will vary but should demonstrate an understanding that the inverse of Equation 2 is a square root equation in which the vertical and horizontal translations are the reverse of the values in the original quadratic equation. Equation 2 is vertically stretched by a factor of 7 while its inverse, the square root equation, is compressed vertically by a factor of $\frac{1}{\sqrt{7}}$.

5. Find the inverse of $y = a\sqrt{(x-h)} + k$.
Possible response is shown on the following page.

6. Find the inverse of $y = a(x-h)^2 + k$.
Possible response is shown on the following page.

Display Transparency B. Ask participants to find the inverse of each equation algebraically.

$$y = a\sqrt{x-h} + k$$

$$x = a\sqrt{y-h} + k$$

$$x - k = a\sqrt{y-h} + k - k$$

$$\frac{x-k}{a} = \frac{a\sqrt{y-h}}{a}$$

$$\left(\frac{x-k}{a}\right)^2 = \left(\sqrt{y-h}\right)^2$$

$$\left(\frac{x+k}{a}\right)^2 = y - h$$

$$\left(\frac{x+k}{a}\right)^2 + h = y - h + h$$

$$y = \left(\frac{x+k}{a}\right)^2 + h \text{ or } y = \left(\frac{1}{a}\right)^2 (x+k)^2 + h$$

$$y = a(x-h)^2 + k$$

$$x = a(y-h)^2 + k$$

$$x - k = a(y-h)^2 + k - k$$

$$\frac{x-k}{a} = \frac{a(y-h)^2}{a}$$

$$\frac{x-k}{a} = (y-h)^2$$

$$\pm\sqrt{\frac{x-k}{a}} = \pm\sqrt{(y-h)^2}$$

$$\pm\sqrt{\frac{x-k}{a}} = y - h$$

$$\pm\sqrt{\frac{x-k}{a}} + h = y - h + h$$

$$\pm\sqrt{\frac{x-k}{a}} + h = y$$

$$y = \frac{\sqrt{x-k}}{\sqrt{a}} + h \text{ or } y = \frac{1}{\sqrt{a}}\left(\sqrt{x-k}\right) + h \text{ OR } y = -\frac{\sqrt{x-k}}{\sqrt{a}} + h \text{ or } y = -\frac{1}{\sqrt{a}}\left(\sqrt{x-k}\right) + h$$

7. Summarize the relationship between *h* and *k* in the square root transformation form and *h* and *k* in the quadratic transformation (vertex) form.
The h and k are reversed. That is, the h in the square root function becomes the k in the quadratic function and vice versa. The k in the square root function becomes the h in the quadratic function and vice versa.

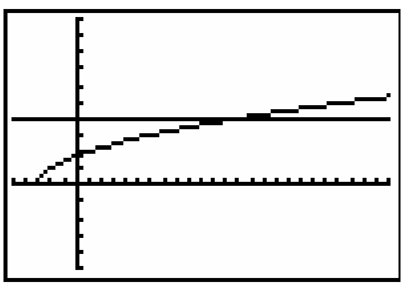
8. Summarize the relationship between *a* in the square root transformation form and *a* in the quadratic transformation (vertex) form.
The "a" in the square root function is inversely proportional to the square of "a" in the quadratic function. The "a" in the quadratic function is inversely proportional to the square root of "a" in the square root function.

<u>Part 7</u>: Solving Square Root Equations and Inequalities

Leader Note:
In this part of the professional development participants investigate solutions to square root equations and inequalities using graphs, tables, and algebraic representations.

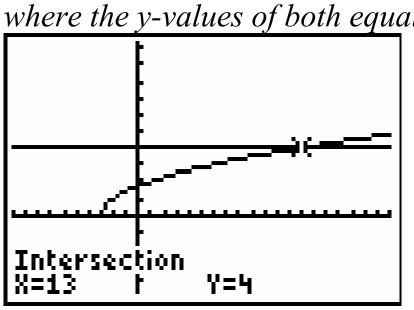
1. Consider the system of equations $y=\sqrt{x+3}$ and $y=4$.

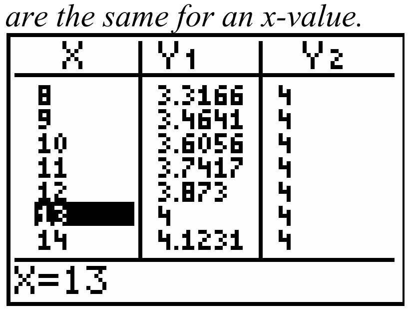
a) Graph the system and sketch the graph. What are the domain and range for each function in this system?



For the function $y=\sqrt{x+3}$, the domain is all real numbers greater than –3, and the range is all y-values greater than 0. For the function $y=4$, the domain is all real numbers, and the range is 4.

b) How can you determine the solution to this system of equations graphically or tabularly?
I can find the point of intersection graphically, and/or I can find the values in the table where the y-values of both equations are the same for an x-value.





X	Y1	Y2
8	3.3166	4
9	3.4641	4
10	3.6056	4
11	3.7417	4
12	3.873	4
13	4	4
14	4.1231	4

X=13

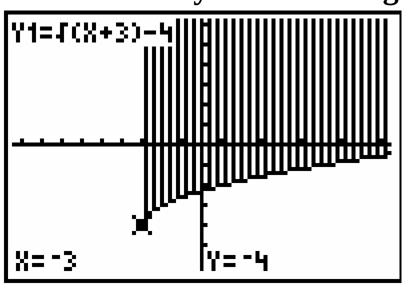
c) What are the coordinates of the point that is a solution for this system?
The solution is (13, 4).

d) How can you use the transitive property to write this system as one equation?
$\sqrt{x+3}=4$

e) How can you solve this equation algebraically?

$$\sqrt{x+3}=4$$
$$\left(\sqrt{x+3}\right)^2=4^2$$
$$x+3=16$$
$$x+3-3=16-3$$
$$x=13$$

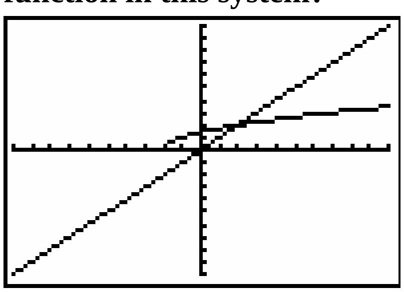
f) What would be your solution set if you had been given $\sqrt{x+3}\geq 4$ **?**



The solution is all real numbers where x is greater than –3 and y is greater than –4.

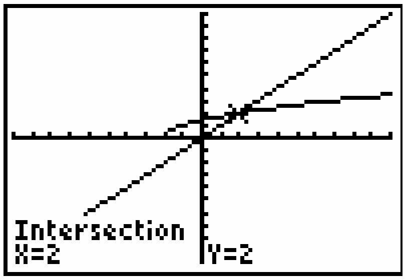
2. Consider the system of equations $y=\sqrt{x+2}$ **and** $y=x$ **.**

a) Graph the system and sketch the graph. What are the domain and range for each function in this system?



For the function $y=\sqrt{x+2}$ ***, the domain is all real numbers greater than –2, and the range is all y-values greater than 0. For the function*** $y=x$ ***, the domain is all real numbers, and the range is all real numbers.***

b) How can you determine the solution to this system graphically or tabularly?
I can find the point of intersection graphically, and/or I can find the values in the table where the y-values of both equations are the same for an x-value.



X	Y1	Y2
-1	1	-1
0	1.4142	0
1	1.7321	1
2	2	2
3	2.2361	3
4	2.4495	4
5	2.6458	5

X=2

c) What are the coordinates of the point that is a solution for this system of equations?
The solution is (2, 2).

d) How can you use the transitive property to write this system as one equation?
$\sqrt{x+2} = x$

e) How can you solve this equation algebraically?

$$\sqrt{x+2} = x$$
$$\left(\sqrt{x+2}\right)^2 = x^2$$
$$x+2 = x^2$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$
$$x = 2 \quad x = -1$$

f) Are both solutions valid? Why or why not?
When I substitute the solutions into the equation, I can verify that 2 is a valid solution. However, –1 produced an extraneous root. It does not give me valid solution.

$$\sqrt{x+2} = x \qquad \sqrt{x+2} = x$$
$$\sqrt{2+2} = 2 \qquad \sqrt{-1+2} = -1$$
$$\sqrt{4} = 2 \qquad \sqrt{1} = -1$$
$$2 = 2 \qquad 1 \neq -1$$

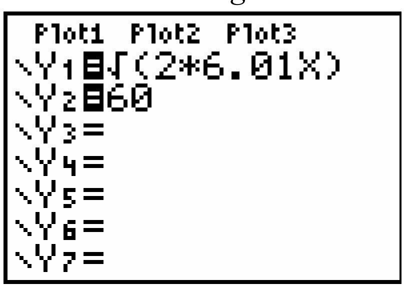
This is also demonstrated graphically and tabularly in ***b)*** *above.*

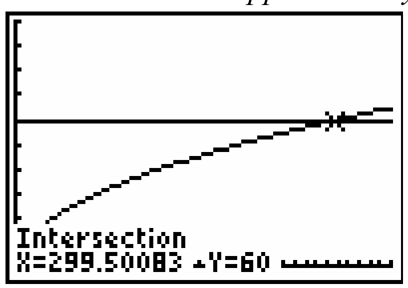
Leader notes:
The purpose of Question 3 is to allow participants to investigate a square root function in a real world context. At the end of the investigation they should be more willing to consider solving square root equations with tables and graphs in addition to the traditional symbolic solution.

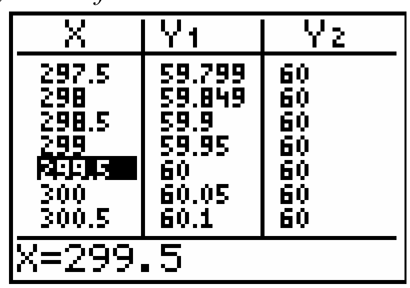
3. Choose one of the following problems. Work with your group to find the solutions(s). Justify your answer. Use chart paper to display your work.

A. Jim is an accident investigator who was asked to determine whether a driver's excessive speed was a factor in a traffic accident. The traditional equation used to determine the speed at which a vehicle was traveling at the onset of the skid is $V_s = \sqrt{2aS_s}$, where *a* is the deceleration force of gravity times friction, and S_s is the length of the skid marks. If the speed limit is 60 mph and the skid marks are 225 ft. long, was the driver exceeding the speed limit? (Use 6.01 for *a*.) What is the maximum length skid mark that would have exonerated the driver? Justify your answer.

No. The driver was traveling at approximately 52 mph.
The maximum length skid mark would have been approximately 299.5 ft.







$$60 = \sqrt{2 * 6.01x}$$

$$60^2 = \left(\sqrt{2 * 6.01x}\right)^2$$

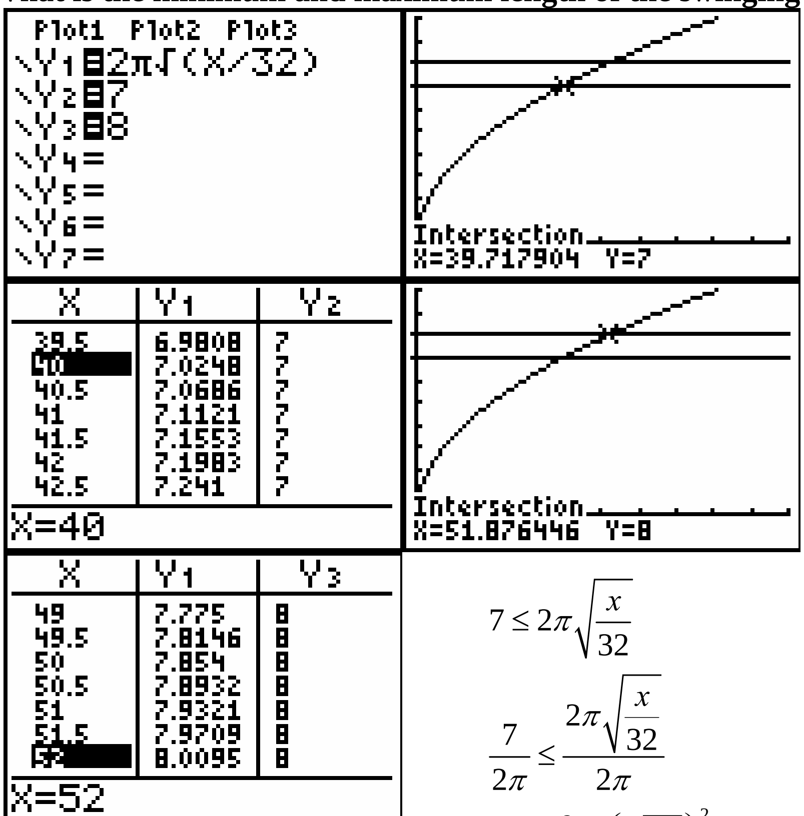
$$3600 = 12.02x$$

$$299.5 \approx x$$

B. At Thalia's favorite amusement park, there is a ride called the "Pirate Ship." People sit in what looks like a huge ship. The "ship" then swings back and forth. Thalia notices that it takes somewhere between 7 and 8 seconds for the ride to make one complete swing back and forth.

> **The function that represents the time in seconds of one complete swing, *t*, based on the height of the swinging bar, *h*, in feet, is** $t = 2\pi\sqrt{\dfrac{h}{32}}$ **.**

What is the minimum and maximum length of the swinging bar?



$$7 \le 2\pi\sqrt{\frac{x}{32}} \qquad\qquad 8 \ge 2\pi\sqrt{\frac{x}{32}}$$

$$\frac{7}{2\pi} \le \frac{2\pi\sqrt{\frac{x}{32}}}{2\pi} \qquad\qquad \frac{8}{2\pi} \ge \frac{2\pi\sqrt{\frac{x}{32}}}{2\pi}$$

$$\left(\frac{7}{2\pi}\right)^2 \le \left(\sqrt{\frac{x}{32}}\right)^2 \qquad\qquad \left(\frac{8}{2\pi}\right)^2 \ge \left(\sqrt{\frac{x}{32}}\right)^2$$

$$(32)\left(\frac{7}{2\pi}\right)^2 \le \frac{x}{32}(32) \qquad\qquad (32)\left(\frac{8}{2\pi}\right)^2 \ge \frac{x}{32}(32)$$

$$39.71790399 \le x \qquad\qquad 51.87644602 \ge x$$

The swinging bar is between 39.71 and 51.88 ft. long.

C. Arnie was taking a picture from the window of his apartment. Unfortunately, he dropped the camera, which landed on the ground at least 2 seconds later. The equation that models the time, *t*, it takes for an object to fall *h* meters is $t = \sqrt{\dfrac{2h}{9.81}}$ **.**

From what height did Arnie drop the camera?

$$2 = \sqrt{\frac{2h}{9.81}}$$

$$2^2 = \left(\sqrt{\frac{2h}{9.81}}\right)^2$$

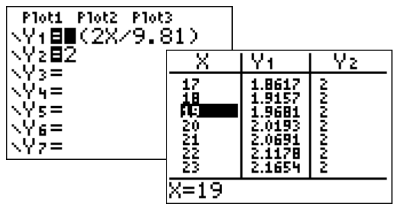
$$4 = \frac{2h}{9.81}$$

$$9.81 \bullet 4 = \frac{2h}{9.81} \bullet 9.81$$

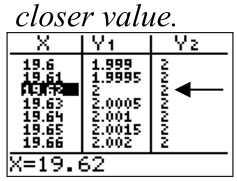
$$39.24 = 2h$$

$$19.62 = h$$

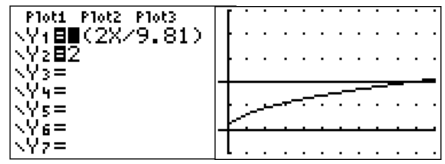
He dropped the camera from a height of 19.62 meters.



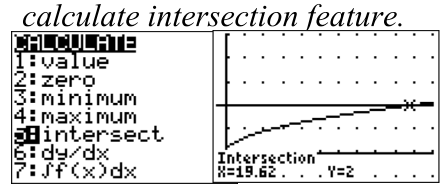
The height is between 19 and 20, so adjust ΔTbl to get a closer value.



He dropped the camera from a height of about 19.62 meters.

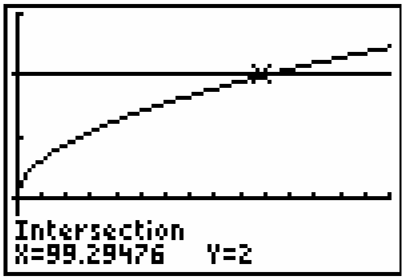


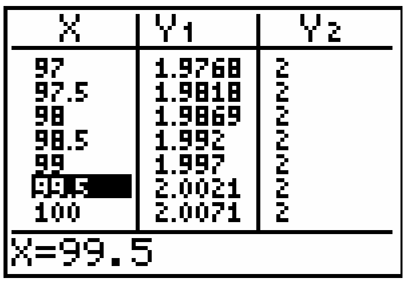
You can trace to the point of intersection, or use the calculate intersection feature.



For a time of 2 seconds, the height is 19.62 meters.

D. Sharon's mother bought a grandfather clock and asked Sharon to determine how long the pendulum must be so the clock keeps accurate time. Sharon found the formula $t = 2\pi\sqrt{\frac{L}{g}}$, where *t* is the time for one complete swing of the clock pendulum, *L* is the length of the pendulum, and *g* is acceleration due to gravity (which is 980 cm/sec^2). Since the time for a complete swing of the pendulum of a grandfather clock must be 2 seconds, how long should the pendulum be?





X	Y1	Y2
97	1.9768	2
97.5	1.9818	2
98	1.9869	2
98.5	1.992	2
99	1.997	2
99.5	2.0021	2
100	2.0071	2

X=99.5

$$t = 2\pi\sqrt{\frac{L}{g}}$$

$$2 = 2\pi\sqrt{\frac{L}{980}}$$

$$\frac{2}{2\pi} = \frac{2\pi\sqrt{\frac{L}{980}}}{2\pi}$$

$$\frac{1}{\pi} = \sqrt{\frac{L}{980}}$$

$$\left(\frac{1}{\pi}\right)^2 = \left(\sqrt{\frac{L}{980}}\right)^2$$

$$\left(\frac{1}{\pi}\right)^2 = \frac{L}{980}$$

$$(980)\left(\frac{1}{\pi}\right)^2 = \frac{L}{980}(980)$$

$$99.29 \approx L$$

The length of the pendulum should be approximately 99.29 cm.

Explain

Leaders' Note: The Maximizing Algebra II Performance (MAP) professional development is intended to be an extension of the ideas introduced in Mathematics TEKS Connections (MTC). Throughout the professional development experience, we allude to components of MTC such as the Processing Framework Model, the emphasis of making connections among representations, and the links between conceptual understanding and procedural fluency.

Debriefing the Experience:

1. **What concepts did we explore in the previous set of activities? How were they connected?**
   Responses may vary. Participants should observe that investigating the physical and graphical representations of square root functions enhances understanding of the concept of square root. Square root equations and inequalities can be solved by graphing the equations and inequalities as systems.

   Facilitation Questions
   - **How would you describe the conceptual progression in this explore phase?**
     Answers may vary. Possible response: We began by exploring a physical representation of the square root function and then progressed to graphical, tabular, and symbolic representations.
   - **What role did representations play in the conceptual progression?**
     Answers may vary. Possible response: We saw the connections among the physical model and the corresponding graphs, tables, and symbolic representations.

2. **What procedures did we use to describe square root functions? How are they related?**
   Tabular, graphical, and symbolic procedures were all used throughout the Explore phase. Ultimately, they are all connected through the numerical relationships used to generate them.

   Facilitation Questions
   - **How would you describe the procedural progression in this explore phase?**
     Answers may vary. Possible response: After collecting the data, we tried several function rules and found one that approximated our data. Then we connected it to both the graph and table. We were able to expand those procedures to form generalizations about transformations and solutions to square root functions.
   - **What role did representations play in the procedural progression?**
     Answers may vary. Possible response: We saw that we could solve square root functions using graphs, tables, and symbolic representations.

3. **What knowledge from Algebra I do students bring about square root functions?**
   Square root is not a student expectation until Algebra II, so students may not have had prior instruction on the square root of a number or square root functions.

4. **After working with square root functions in Algebra II, what are students' next steps in Precalculus or other higher mathematics courses?**
   According to the Precalculus TEKS, students will expand their understanding of square root functions. They will be expected to apply basic transformations and compositions with square root functions, including $\left|f\left(x\right)\right|$, *and* $f\left(\left|x\right|\right)$, *to the parent functions.*

Anchoring the Experience:

5. **Distribute to each table group a poster-size copy of the Processing Framework Model.**

6. **Ask each group to respond to the question:**

   Where in the processing framework would you locate the different activities from the Explore phase?

7. **Participants can use one color of sticky notes to record their responses. In future Explain phases, participants will use other colors of sticky notes to record their responses.**

Horizontal Connections within the TEKS

8. **Direct the participants' attention to the second layer in the Processing Framework Model: Horizontal Connections among Strands.**

9. **Prompt the participants to study the Algebra II TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

10. **Invite each table group to share two connections that they found and record them so that they are visible to the entire group.**

Vertical Connections within the TEKS

11. **Direct the participants' attention to the third layer in the Processing Framework Model: Vertical Connections across Grade Levels.**

12. **Prompt the participants to study the Algebra I, Geometry, Math Models, and Precalculus TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

13. **Invite each table group to share two connections that they found, recording so that the entire large group may see.**

14. **Provide each group of participants with a clean sheet of chart paper. Ask them to create a "mind map" for the mathematical term of "square root function."**
    See next page for possible response.

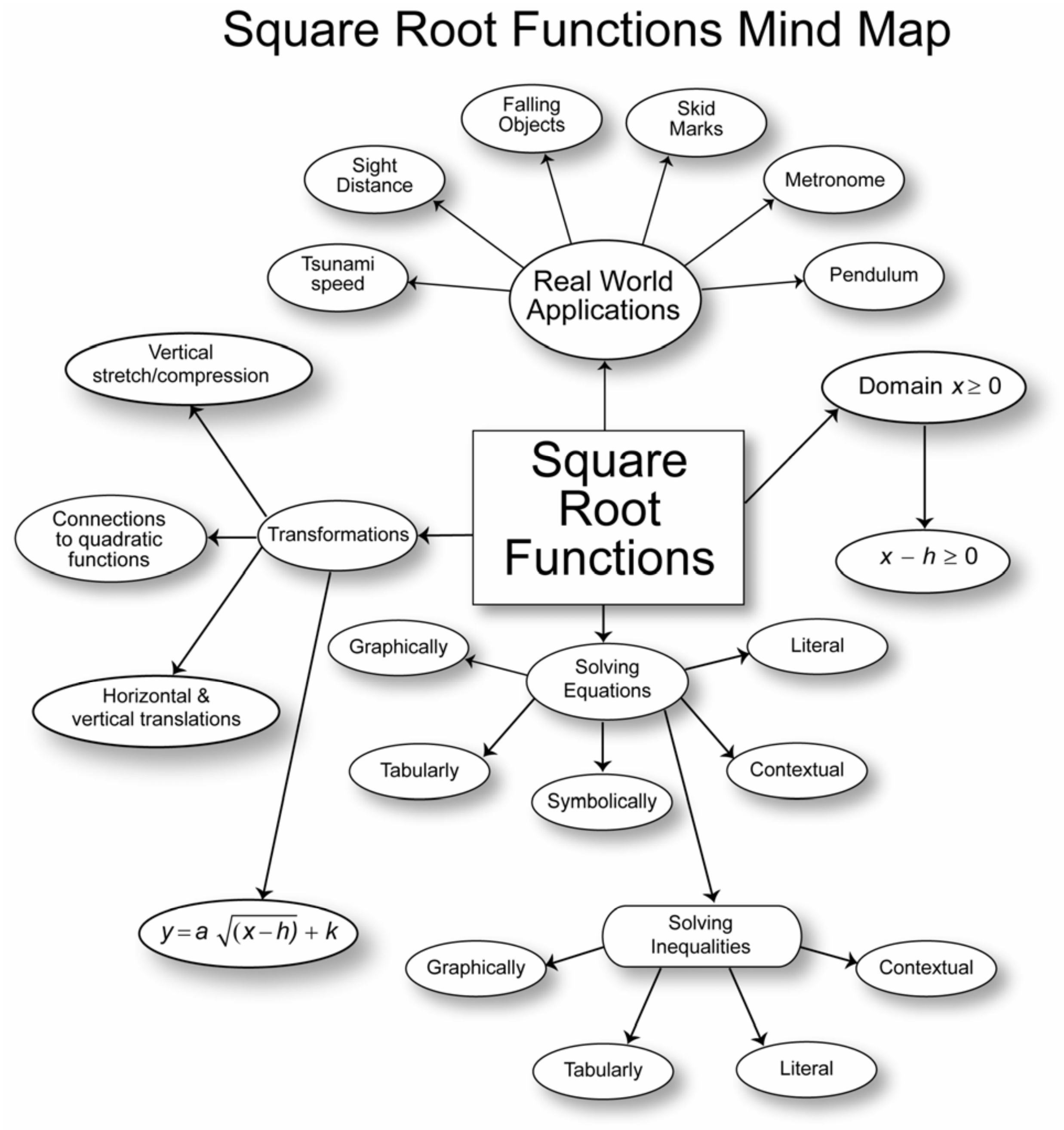
15. **Provide an opportunity for each group to share their mind maps with the larger group. Discuss similarities, differences, and key points brought forth by participants.**

16. **Distribute the vocabulary organizer template to each participant. Ask participants to construct a vocabulary model for the term "square root functions."**

17. **When participants have completed their vocabulary models, ask participants to identify strategies from their experiences so far in the professional development that could be used to support students who typically struggle with Algebra II topics.**

    Note to Leader: *You may wish to have each small group brainstorm a few ideas first, then share their ideas with the large group while you record their responses on a transparency or chart paper.*

18. **How would this lesson maximize student performance in Algebra II for teaching and learning the mathematical concepts and procedures associated with square root functions?**
    Responses may vary. Anchoring procedures within a conceptual framework helps students understand what they are doing so that they become more fluent with the procedures required to accomplish their tasks. Problems present themselves in a variety of representations; providing students with multiple procedures to solve a given problem empowers students to solve the problem more easily.


Square Root Functions Mind Map
Square Root Functions
Real World Applications
Tsunami speed
Sight Distance
Falling Objects
Skid Marks
Metronome
Pendulum
Domain $x \geq 0$
$x - h \geq 0$
Transformations
Vertical stretch/compression
Connections to quadratic functions
Horizontal & vertical translations
$y = a\sqrt{(x-h)} + k$
Solving Equations
Graphically
Literal
Tabularly
Symbolically
Contextual
Solving Inequalities
Graphically
Contextual
Tabularly
Literal

Elaborate

Leaders' Note: *In this phase, participants will extend their learning experiences to their classroom.*

1. **Distribute the 5E Student Lesson planning template. Ask participants to think back to their experiences in the Explore phase. Pose the following task:**

   What might a student-ready 5E lesson on square root functions look like?
   - **What would the Engage look like?**
   - **Which experiences/activities would students explore firsthand?**
   - **How would students formalize and generalize their learning?**
   - **What would the Elaborate look like?**
   - **How would we evaluate student understanding of inverses of relations/functions?**

2. **After participants have recorded their thoughts, direct them to the student lesson for square root functions. Allow time for participants to review lessons.**

3. **How does this 5E lesson compare to your vision of a student-centered 5E lesson?**
   Responses may vary.

4. **How does this lesson help remove obstacles that typically keep students from being successful in Algebra II?**
   By connecting the concept of square root to a physical model, students gain a better understanding of the square root function and the limitations of using a linear function to describe the model. By solving square root equations as systems and through graphing, solutions can be verified and connected to the algebraic processes commonly used to find those solutions. Students can use alternate methods with which to solve meaningful problems.

5. **How does this lesson maximize your instructional time and effort in teaching Algebra II?**
   Taking time to create a solid conceptual foundation reduces the need for re-teaching time and effort and increases student participation in the learning process. Conceptual connections to algebraic process strengthen the understanding of square root functions and mirror the links among multiple representations.

6. **How does this lesson maximize student learning in Algebra II?**
   Using multiple representations and foundations for functions concepts allows students to make connections among different ideas. These connections allow students to apply their learning to new situations more quickly and readily.

7. **How does this lesson accelerate student learning and increase the efficiency of learning?**
   Foundations for functions concepts such as function transformations transcend all kinds of functions. A basic toolkit for students to use when working with functions allows students to rethink what they know about linear and quadratic functions while they are learning concepts and procedures associated with other function families.

8. **Read through the suggested strategies on Strategies that Support English Language Learners. Consider the possible strategies designed to increase the achievement of English language learners.**
   As participants read through the strategies that support English language learners and strategies that support students with special needs, they may notice that eight of the ten strategies are the same. The intention is to underscore effective teaching practices for all students. However, English language learners have needs specific to language that students with special needs may or may not have. The two strategies that are unique to the English language learners reflect an emphasis on language. Students with special needs may have prescribed modifications and accommodations that address materials and feedback. Students with special needs often benefit from progress monitoring with direct feedback and adaptation of materials for structure and/or pacing. A system of quick response is an intentional plan to gather data about a student's progress to determine whether or not the modification and (or) accommodation are (is) having the desired effect. The intention of the strategies is to provide access to rigorous mathematics and support students as they learn rigorous mathematics.

9. **What evidence of these strategies do you find in this portion of the professional development?**
   Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening. This contributes to the teacher's ability to create an emotionally safe environment for learning. Tools such as the CBR and graphing calculator can be used to communicate about and solve problem situations involving square root functions.*

10. **Which strategies require adaptation of the materials in this portion of the professional development?**
    Responses may vary. Most of the strategies are incorporated throughout the materials.

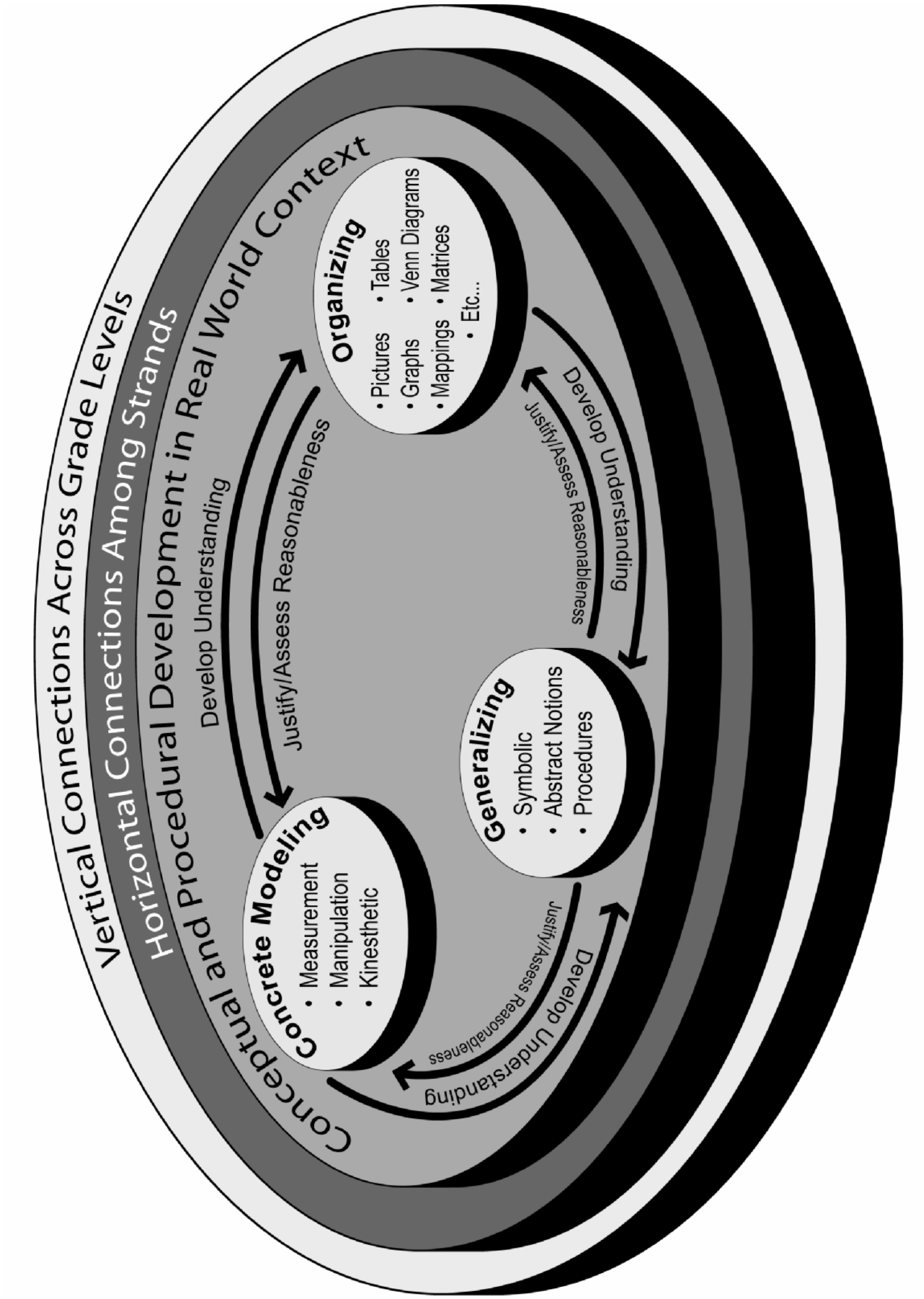
11. **Read through the suggested strategies on Strategies that Support Students with Special Needs. Consider the possible strategies designed to increase the achievement of students with special needs.**

12. **What evidence of these strategies do you find in this portion of the professional development?**
    Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening. This contributes to the teacher's ability to create an emotionally safe environment for learning. Tools such as the CBR and graphing calculator can be used to communicate about and solve problem situations involving square root functions.*

13. Which strategies require adaptation of the materials in this portion of the professional development?

Responses may vary. Most of the strategies are incorporated throughout the materials. Some materials may need to be modified for format or structure.

Processing Framework Model



Vocabulary Organizer

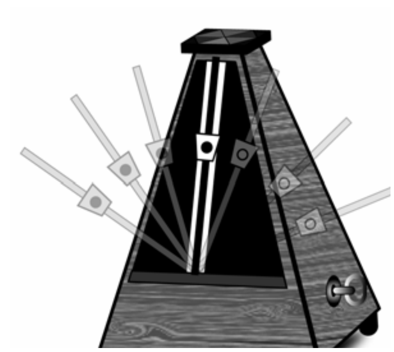
My definition	**Personal Association**
Example	**Non-Example**

Sample vocabulary organizer for square root functions

My definition

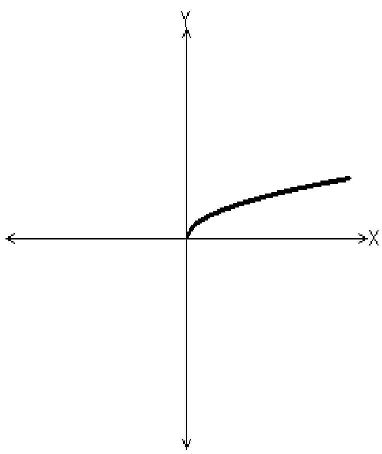
The inverse of the positive side of the quadratic parent function

Personal Association

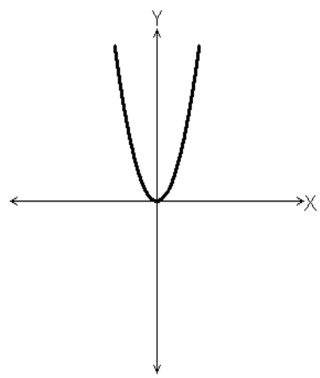


Square root function

Example



Non-Example



5E Student Lesson Planning Template

Description	Activity
Engage The activity should be designed to generate student interest in a problem situation and to make connections to prior knowledge. The instructor initiates this stage by asking meaningful questions, posing a problem to be solved, or by showing something intriguing.	
Explore The activity should provide students with an opportunity to become actively involved with the key concepts of the lesson through a guided exploration requiring them to probe, inquire, and question. The instructor actively monitors students as they interact with each other and the activity.	
Explain Students collaboratively begin to sequence events/facts from the investigation and communicate these findings to each other and the instructor. The instructor, acting in a facilitation role, formalizes student findings by providing further explanations and additional meaning or information, such as correct terminology.	
Elaborate Students extend, expand, or apply what they have learned in the first three stages and connect this knowledge with prior learning to deepen understanding. Instructors can use the Elaborate stage to verify students' understandings.	
Evaluate Evaluation occurs throughout students' learning experiences. More formal evaluation can be conducted at this stage. Instructors can determine whether the learner has reached the desired level of understanding the key ideas and concepts.	

Strategies that Support English Language Learners (ELL)

Strategy	Explore, Explain, Elaborate 2
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Be conscious of tone and diction. Speak slowly and distinctly.	
Incorporate language skills (reading, writing, speaking, and listening) into instruction.	

Strategies that Support Students with Special Needs

Strategy	**Explore, Explain, Elaborate 2**
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Use a system of quick response to needs and accommodations including progress monitoring to inform instruction.	
Accommodate materials for format, structure, sequence, etc. as needed.	

Transparency A

Equation 1

$$y = 3\sqrt{x - 5} + 6$$

Equation 2

$$y = 7(x - 2)^2 + 4$$

Transparency B

$$y = a\sqrt{x-h} + k$$

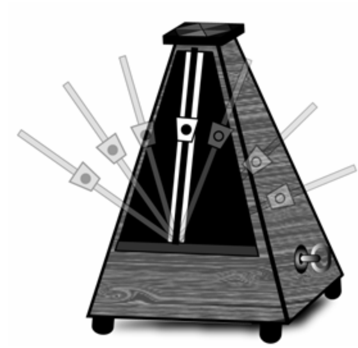
$$y = a(x-h)^2 + k$$

Participant Pages: Square Root Functions

Explore

Part 1: Setting the Stage

A metronome frequently is used in music to mark exact time using a repeated tick. The frequency of the ticks varies in musical terms from slow *(largo,* about 40-60 beats per minute) to fast *(presto,* 168-208 beats per minute). Individual instrumentalists, choirs, bands, and orchestras all use metronomes to ensure that the beat of the music is consistent with the instructions of the composer and does not unintentionally speed up or slow down while the piece is being played.



What do you notice about the frequency of the ticks and setting of the weight as the music is being played?

Part 2: Modeling and Gathering Data

Divide participants into groups of 4. Each person in the group has a job.

Materials manager: Gets the necessary materials, directs the team in setting up the investigation, holds the pencil with the suspended bottle, and shortens the spring when needed.

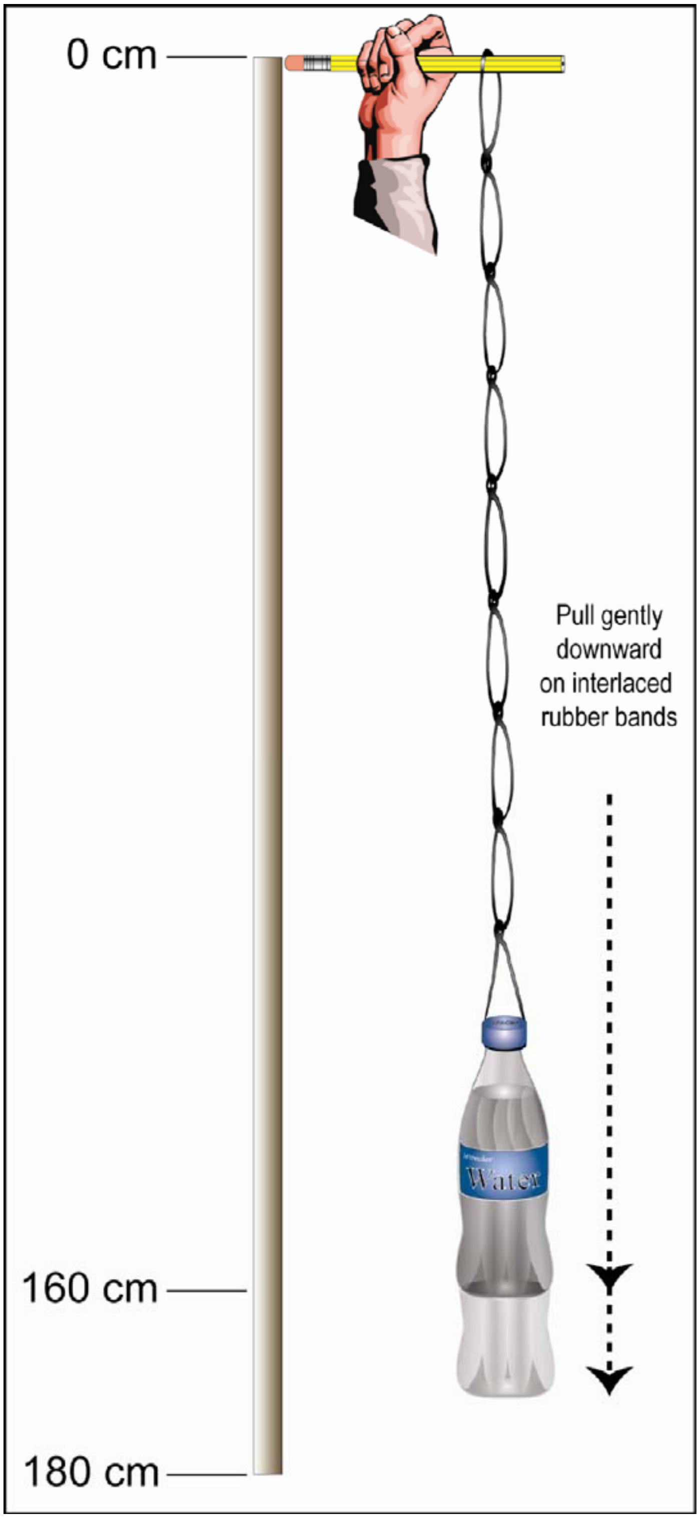
Measures manager: Measures the distance from the pencil to the bottom of the bottle for each length, initiates the bounce by pulling the suspended bottle down an additional 10 centimeters and counts the bounces (10 at each height).

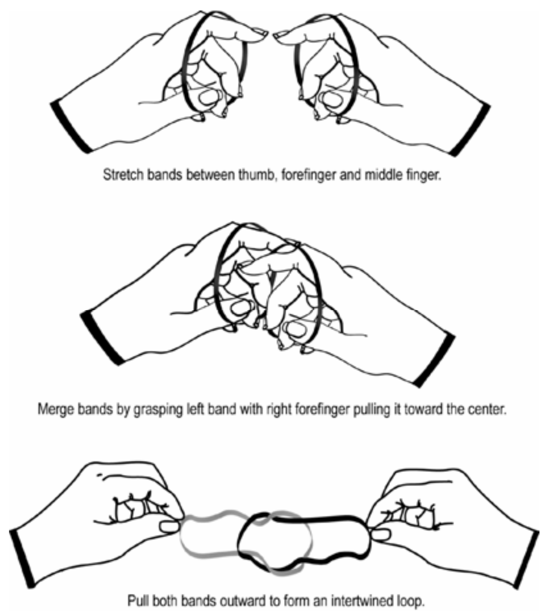
Time manager: Uses a stop watch to determine the length of each 10-bounce period of time. The time starts when the bottle is released by the measures manager and ends when the bottle completes its 10^{th} bounce.

Data manager: Records the necessary measurements in the table and shares the data with the team.

Set-up Instructions

Step 1. The materials manager should get the necessary materials and ask two of the team members to secure the tape measure or meter sticks against the wall. The tape measure or meter sticks should be positioned perpendicular to the floor so that the "zero end" is at 180-200 centimeters above the floor.





Step 2. While the tape measure or meter sticks are being positioned, the materials manager and remaining team member(s) build the rubber band spring by looping rubber bands together until the length of the spring is about 1 meter. This task will go more quickly if each person makes about half of the spring. Then the pieces can be joined.

Step 3. Secure one end of the rubber band spring to the pencil and the other around the neck of the bottle. It also works to remove the cap, insert the end of the spring in the bottle, and screw the cap back on.

Step 4. The measures manager secures the spring so that it is approximately 160 centimeters in length. After the bottle remains motionless for a few seconds, he should measure the actual length of the spring. The length of the spring includes the length of the rubber band and the length of the bottle.

Step 5. The measures manager pulls the bottle downward about 10 cm and releases it.

Step 6. The time manager starts the stopwatch when it is released and stops it at the end of 10 complete bounces. It may be helpful to have all team members count aloud together.

Step 7. The data manager records the number of seconds in the table under Trial 1 for 160 cm. *Hint: It is more meaningful to start with the spring fully extended and to shorten the spring than to begin at the top and work down.*

Step 8. Repeat for Trials 2 and 3. Average the data from the 3 trials and record in the Average Time column.

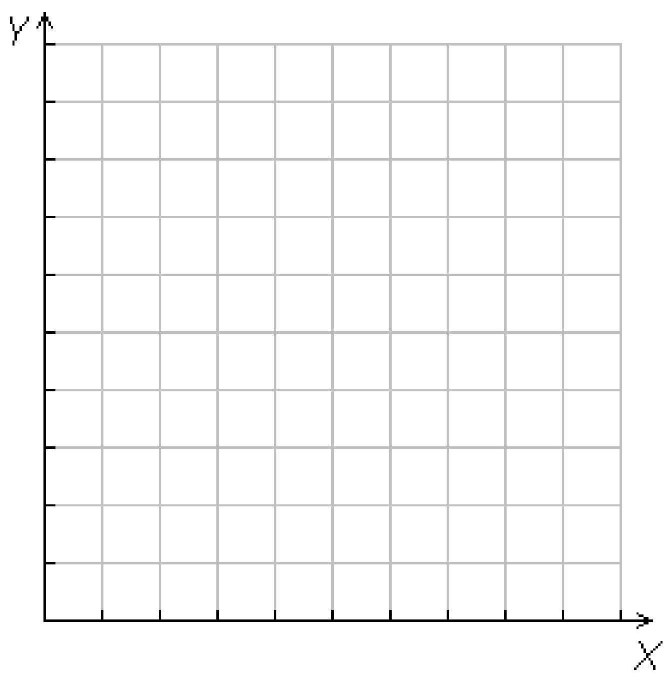
Step 9. The materials manager who is holding the pencil shortens the spring by wrapping it around the pencil until the desired length of 140 is obtained.

Step 10. Continue repeating the procedure with shortened lengths of rubber band spring. Continue to record your data.

1. Record your data in the table below.

Approximate Length of Spring (cm) *x*	Actual Length of Spring (cm) *x*	Trial 1	Trial 2	Trial 3	Average Time *y*
0					
20					
30					
40					
60					
80					
100					
120					
140					
160					

2. Make a scatterplot of the data you collected.



3. What is the independent variable?

4. What is the dependent variable?

5. Write a dependency statement relating the two variables.

6. What is a reasonable domain for the set of data?

7. What is a reasonable range for the set of data?

8. Is this data set continuous or discrete? Why?

9. Does the set of data represent a function? Why?

10. Write a summary statement about what happened in this data investigation.

11. Is the function increasing or decreasing?

12. Is the rate of change constant?

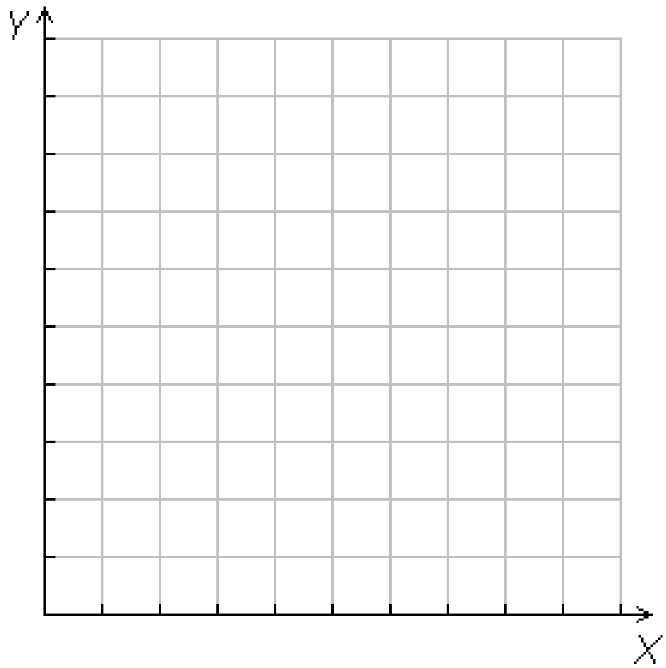
13. Does the data you collected appear to be a linear, quadratic, exponential, or some other type of parent function? Why?

14. How is the bottle bounce activity similar to the ticks of a metronome?

15. What kind of function do you think models the ticking of the metronome? Why?

<u>Part 3</u>: Analyzing the Data

1. How can you determine whether this function is the inverse of another parent function using patty paper?



2. Input your values into L_1 and L_2 of a graphing calculator, letting L_1 be independent values and L_2 dependent values, and create a scatterplot of the original data. Sketch your graph.

3. Create a second scatterplot that represents an inverse of the data. Use a different plot symbol for this scatterplot. Determine a new domain and range, and set a new viewing window. Sketch your graph.

4. What changes must you make to the window to view the second set of data?

5. Which parent function do the *reflected* points most closely appear to represent?

6. How did you determine your function?

7. How might your confirm your conjecture?

8. Without using regression, find a function that approximates your data for Plot 2.

9. Does your viewing window allow you to see both sides of the parabola? If not, readjust your viewing window. Sketch your graph.

10. How could you use this function to find a function that would approximate the first scatterplot you graphed?

11. Reset your window to view Plot 1. Enter the equation in the equation editor. Is your graph a close fit to the data in Plot 1? Sketch your graph.

12. Compare and contrast the graphs of a quadratic function and a square root function. How are they similar, and how are they different?

13. Why are there no negative coordinates in the square root function?

14. What is the domain of a square root function?

15. What is the range of a square root function?

16. What conclusions can you make about the attributes of a square root function?

17. What conclusions can you make about the collected data?

<u>Part 4</u>: Making Symbolic Generalizations

Transformations of Square Root Functions Card Sort

1. Place the cards in the proper row and column.

Description	Example	Example	Notation
Vertical Translation Up			
Vertical Translation Down			
Horizontal Translation Left			
Horizontal Translation Right			
Vertical Stretch			
Vertical Compression			
Reflection			

2. Describe the role of a.

3. Describe the role of h.

4. Describe the role of k.

5. Using x, a, h, and k, write an equation that could be used to summarize the transformations to the square root function.

6. Revisiting the bottle bounce investigation, describe the transformation to the square root parent function that represents your data.

<u>Part 5 (Optional Extension)</u>: Investigating the Coefficient of *x*.

1. Using $f(x) = a\sqrt{\frac{1}{b}(x-h)} + k$, predict the changes in the parent function for the following functions. Then check with your graphing calculator.

   a. $f(g) = \sqrt{-x}$

   b. $f(g) = \sqrt{-3x} + 4$

   c. $f(g) = \sqrt{\frac{1}{2}(x-3)}$

   d. $f(g) = 2\sqrt{-\frac{1}{3}(x+4)} - 5$

2. What can you summarize about transformations of the square root parent function as a result of changes to $\frac{1}{b}$?

<u>Part 6 (Optional Extionsion)</u>: Connecting the Roles of *a, h,* and *k* in Square Root and Quadratic Functions

Equation 1 $$y = 3\sqrt{x-5} + 6$$

Equation 2 $$y = 7(x-2)^2 + 4$$

1. Find the inverse of Equation 1.

2. Numerically and graphically compare and contrast Equation 1 and its inverse.

3. Find the inverse of Equation 2.

4. Numerically and graphically compare and contrast Equation 2 and its inverse.

5. Find the inverse of $y = a\sqrt{(x-h)} + k$.

6. Find the inverse of $y = a(x-h)^2 + k$.

7. Summarize the relationship between h and k in the square root transformation form and h and k in the quadratic transformation (vertex) form.

8. Summarize the relationship between a in the square root transformation form and a in the quadratic transformation (vertex) form.

<u>Part 7</u>: Solving Square Root Equations and Inequalities

1. Consider the system of equations $y = \sqrt{x+3}$ and $y = 4$.
   a) Graph the system and sketch the graph. What are the domain and range for each function in this system?

   b) How can you determine the solution to this system of equations graphically or tabularly?

   c) What are the coordinates of the point that is a solution for this system?

   d) How can you use the transitive property to write this system as one equation?

   e) How can you solve this equation algebraically?

   f) What would be your solution set if you had been given $\sqrt{x+3} \geq 4$?

2. Consider the system of equations $y = \sqrt{x+2}$ and $y = x$.

a) Graph the system and sketch the graph. What are the domain and range for each function in this system?

b) How can you determine the solution to this system of equations graphically or tabularly?

c) What are the coordinates of the point that is a solution for this system?

d) How can you use the transitive property to write this system as one equation?

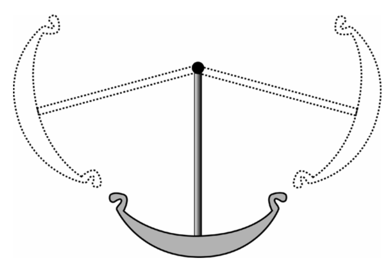
e) How can you solve this equation algebraically?

f) Are both solutions valid? Why or why not?

3. Choose one of the following problems. Work with your group to find the solutions(s). Justify your answer. Use chart paper to display your work.

   A. Jim is an accident investigator who was asked to determine whether a driver's excessive speed was a factor in a traffic accident. The traditional equation used to determine the speed at which a vehicle was traveling at the onset of the skid is $V_s = \sqrt{2aS_s}$, where *a* is the deceleration force of gravity times friction, and S_s is the length of the skid marks. If the speed limit is 60 mph and the skid marks are 225 ft. long, was the driver exceeding the speed limit? (Use 6.01 for *a*.) What is the maximum length skid mark that would have exonerated the driver? Justify your answer.

   B. At Thalia's favorite amusement park, there is a ride called the "Pirate Ship". People sit in what looks like a huge ship. The "ship" then swings back and forth. Thalia notices that it takes somewhere between 7 and 8 seconds for the ride to make one complete swing back and forth. What is the minimum and maximum length of the swinging bar?



The function that represents the time in seconds of one complete swing, *t*, based on the height of the swinging bar, *h*, in feet, is $t = 2\pi\sqrt{\frac{h}{32}}$.

C. Arnie was taking a picture from the window of his apartment. Unfortunately, he dropped the camera, which landed on the ground at least 2 seconds later. The equation that models the time, t, it takes for an object to fall h meters is $t = \sqrt{\frac{2h}{9.81}}$. From what height did Arnie drop the camera?

D. Sharon's mother bought a grandfather clock and asked Sharon to determine how long the pendulum must be so the clock keeps accurate time. Sharon found the formula $t = 2\pi\sqrt{\frac{L}{g}}$, where t is the time for one complete swing of the clock pendulum, L is the length of the pendulum, and g is acceleration due to gravity (which is 980 cm/sec^2). Since the time for a complete swing of the pendulum of a grandfather clock must be 2 seconds, how long should the pendulum be?

Leader Notes: Distance to the Fire Station

Purpose:

The participants will model the absolute value of a number through The Fire Station Problem and an absolute value function through the use of a CBR in The Relay Race. They will explore the effects of parameter changes on the absolute value parent function. They will make connections among the characteristics of an absolute value function and solutions to absolute value equations and inequalities.

Descriptor:

This phase has four parts. In the first part, participants will graph the distances from a fire station to other buildings on the same street to develop the concept of absolute value. In the second part, participants will explore absolute value functions concretely using the Calculator-Based Ranger to model the graph and properties of an absolute value function and numerically by examining the number patterns found in tables of data values. In the third part, participants will examine the effects of parameter changes on the vertex form of absolute value functions. In the fourth part, participants will make connections among the algebraic processes of solving absolute value equations and the graphs and functions related to those equations.

Duration:

2.5 hours

TEKS:

a5 **Tools for algebraic thinking**. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 **Underlying mathematical processes.** Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.1 **Foundations for functions.** The student uses properties and attributes of functions and applies functions to problem situations.

2A.1A The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

2A.1B The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

2A.2 **Foundations for functions.** The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

2A.2A The student is expected to use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

2A.3 **Foundations for functions.** The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the solutions.

2A.3A The student is expected to analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

2A.3B The student is expected to use algebraic methods, graphs, tables, or matrices to solve systems of equations or inequalities.

2A.4 **Algebra and geometry.** The student connects algebraic and geometric representations of functions.

2A.4A The student is expected to identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x (($f(x) = \sqrt{x}$), and reciprocal of x $\left(f(x) = \dfrac{1}{x} \right)$.

2A.4B The student is expected to extend parent functions with parameters such as a in $f(x) = \dfrac{a}{x}$ and describe the effects of the parameter changes on the graph of parent functions.

TAKS™ Objectives Supported:

While the Algebra II TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 3: Linear Functions
- Objective 4: Linear Equations and Inequalities
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance: Copies of participant pages, copies of **The Fire Station Problem Graphic Sheets** taped together

Presenter Materials: Overhead graphing calculator, CBR

Per group: **The Fire Station Problem** G**raphic Sheets** taped together to represent a street, CBR, link cable, chart paper

Per participant: Copy of participant pages, graphing calculator

Explore

<u>Part 1:</u> The Fire Station Problem

Leader Notes:
In this part of the professional development, participants investigate the concept of the absolute value of a number by locating a building a specific distance from the fire station. The fire station graphic sheets must be taped together so that the fire station is in the center of the street and the car is on the left. Explain to participants that the graphic on their tables represent the main street in a city. Ask participants to read the opening paragraph of the fire station Problem and answer the questions that follow on their participant pages. At the end of Part 1, participants will be able to connect a context for the absolute value of a number to a graphical representation using linear functions and compare linear equations to the graph of an absolute value situation.

> **A fire station is located on Main Street and has buildings at every block to the right and to the left. You will investigate the relationship between the address number on a building and its distance in feet from the fire station. On average, a mile in the city is composed of 16 city blocks. So each city block is about 330 feet long (5280 feet ÷ 16 = 330 feet). Each building is centered on the block.**

1. **Complete the table below that relates the address of a building (*x*) with its distance in feet from the fire station (*y*).**

Address Number (*x*)	Distance in Feet from the Fire Station (*y*)
800	*1320*
900	*990*
1000	*660*
1100	*330*
1200	*0*
1300	*330*
1400	*660*
1500	*990*
1600	*1320*

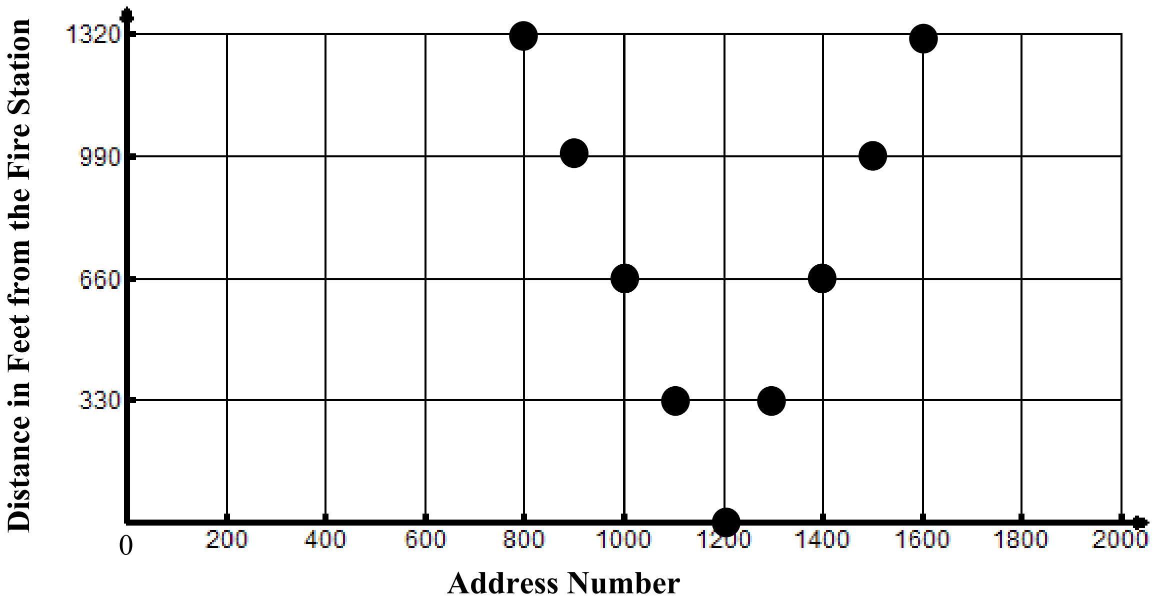
2. **Which building is 660 feet way from the fire station? Explain your answer.**
There are two buildings that are 660 feet from the fire station: the church and the ice cream shop. The church is 660 feet to the left of the fire station (as you look towards the fire station), and the ice cream shop is 660 feet to the right of the fire station.

3. **If we send someone to the building that is 660 feet away from the fire station, how will she know that she has arrived at the correct place?**
She will not know she has arrived at the correct place unless we tell her which direction to walk, or we indicate whether the building is to the right or the left of the fire station.

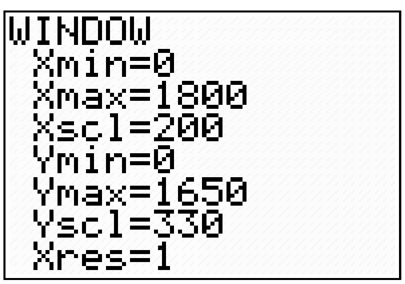
4. **What words might we use to describe two locations that are the same distance from the fire station?**
We describe them in terms of direction (north, south, east, west, 12 o'clock, 1 o'clock, etc.).

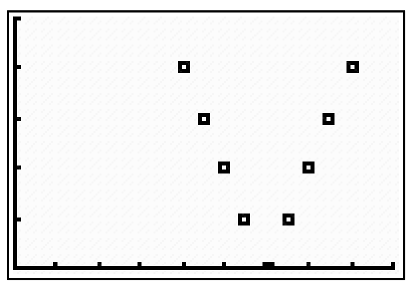
5. **Suppose the buildings on Main Street are renumbered as if they are on a number line so that the location of the fire station represents 0. How do we describe two numbers with the same distance from 0?**
We say that the two numbers have the same absolute value.

6. Draw a scatterplot that represents the data in the table.



7. Make a scatterplot of your data using your graphing calculator. Describe your viewing window.
Responses may vary. Possible responses are shown below.





8. What function or functions might the students use to describe the scatterplot?
Responses may vary. Students may use linear functions to describe the data. Students may also try to use a quadratic function to describe the data.

Leader note:
Algebra II teachers may assume that students have had prior instruction in absolute value. However, the concept of absolute value does not explicitly appear in the TEKS until Algebra II.

9. Find two linear functions that pass through the data points. What process did you use to find the equations of the lines?
The linear functions are $y = -3.3x + 3960$ and $y = 3.3x - 3960$. Possible processes may include table-building to find the y-intercepts, use of the slope formula to find the slope, use of point-slope, and use slope-intercept form. Possible approaches are described in detail on the next three pages.

First, we can find the equation of the line that passes through (800, 1320) and (1200, 0).

From the data table we can choose two points between (800, 1320) and (1200, 0), such as (900, 990) and (1000, 660), and use the formula for slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{660 - 990}{1000 - 900}$$

$$m = \frac{-330}{100}$$

$$m = -3.3$$

To find the y-intercept, we have three options: work backward in the table from (1200, 0) to the point (0, b) on the positive y-axis, use point-slope, or substitute a point and the slope into y = mx + b.

Option 1: Work backward in the table from (1200, 0) to the point (0, b) on the positive y-axis.

x	y
0	3960
100	3630
200	3300
300	2970
400	2640
500	2310
600	1980
700	1650
800	1320
900	990
1000	660
1100	330
1200	0

The y-intercept is 3960. Therefore, one equation is y = –3.3x + 3960.

Option 2: Using point-slope to find the y-intercept. We can choose the point (900, 990).

$$y - y_1 = m(x - x_1)$$
$$y - 990 = -3.3(x - 900)$$
$$y - 990 = -3.3x + 2970$$
$$y = -3.3x + 3960$$

Option 3: Substituting a point, such as (900, 990), and the slope into slope-intercept form and solving for b.

$$y = mx + b$$
$$990 = -3.3(900) + b$$
$$990 = -2970 + b$$
$$3960 = b$$
$$y = -3.3x + 3960$$

Now, we can find the equation of the line that passes through the points (1200, 0) and (1600, 1320).

From the data table we can choose two points between (1200, 0) and (1600, 1320), such as (1300, 330) and (1400, 660), and use the formula for slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{660 - 330}{1400 - 1300}$$
$$m = \frac{330}{100}$$
$$m = 3.3$$

To find the y-intercept, we have three options: work backward in the table from (1200, 0) to the point (0, b) on the negative y-axis, use point-slope, or substitute a point and the slope into y = mx + b.

Option 1: Work backward in the table from (1200, 0) to the point (0, b) on the negative y-axis.

x	*y*
0	–3960
100	–3630
200	–3300
300	–2970
400	–2640
500	–2310
600	–1980
700	–1650
800	–1320
900	–990
1000	–660
1100	–330
1200	0

The y-intercept is –3960. Therefore, the second equation is y = 3.3x – 3960.

Option 2: Using point-slope to find the y-intercept. We can choose the point (900, –990).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-990) &= 3.3(x - 900) \\ y + 990 &= 3.3(x - 900) \\ y - 990 &= 3.3x - 2970 \\ y &= 3.3x - 3960 \end{aligned}$$

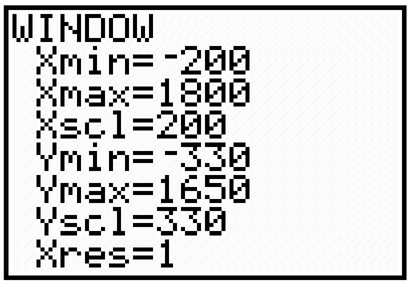
Option 3: Substituting a point, such as (900, –990), and the slope into slope-intercept form and solving for b.

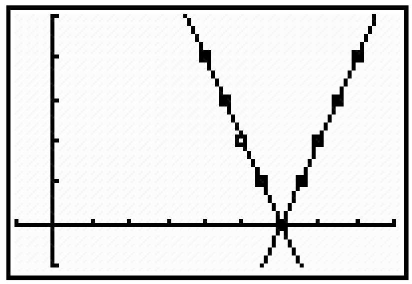
$$\begin{aligned} y &= mx + b \\ -990 &= 3.3(900) + b \\ -990 &= 2970 + b \\ -3960 &= b \\ y &= 3.3x - 3960 \end{aligned}$$

10. Graph the equations on your calculator. How are the equations similar? How are they different?

The equations are similar because they each have a constant rate of change. The absolute value of the slopes of the equations is 3.3. The absolute value of the y-intercepts of the equations is 3960. The equations are different because the slope of one equation is –3.3 and the slope of the other is 3.3. The y-intercept of one equation is 3960 and the y-intercept of the other is –3960.

11. If necessary adjust the window to clearly see the intersection of the two lines. What does the intersection of these two lines represent? Sketch the graph.





The intersection of the two lines represents the location of the fire station.

12. Where do the equations fit the graph of the data points? Where do the equations not fit the graph of the data points?

The equations fit the data points above the x-axis. The equations do not fit the data points below the y-axis.

13. How well do the linear functions model the data points?

Linear functions model the data points above the point of intersection well. Below the point of intersection, the functions do not fit the data.

14. Write summary statements about the conceptual understanding of the absolute value of a number and linear equations that model a situation using absolute value.

Responses may vary. The conceptual understanding of the absolute value of a number can be modeled through a situation relating address number and distance from a particular location. Linear equations can be used to model an absolute value function, but contain points not in the absolute value function.

<u>Part 2</u>: The Relay Race

Leader Notes:
In this part of the professional development, participants model the concept of the absolute value function by walking toward and away from a motion detector at a constant speed. At the end of this part, participants should be able to model an absolute value function using the CBR and connect the graph of the model to an absolute value function.

Technology Tip:
Participants may not have experience using the CBR. Be prepared to assist participants who have questions regarding the data collection process. You may need to explain to the participants that the Calculator-Based Ranger motion detector (CBR) measures the distance between an object and itself at a specific point in time.

Facilitation Tip:
You will want to have someone read aloud the problem situation at the top of the participant page. You will also want to ask a volunteer to model the relay race before the rest of the group. Do not show them the graph of the function.

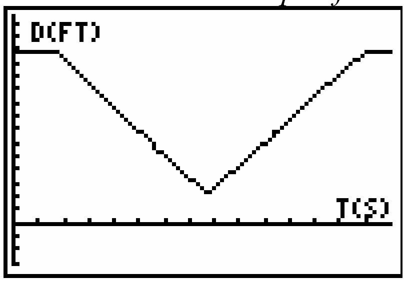
> **Pretend you are participating in an unusual relay race. The object of the race is to walk toward the CBR at a slow steady rate as if to pick up something and then immediately to walk backwards away from the CBR at the same rate without stopping. The person whose rate walking towards the CBR matches the rate walking away from the CBR and who changes direction instantly wins the race!**

Note to Leader: Participants may notice that their rates are not exactly additive inverses of each other. The difference may occur from people's tendency to walk more slowly when they are walking backwards. You may want to suggest that groups collect new data when they actually turn around quickly so that they are walking forward.

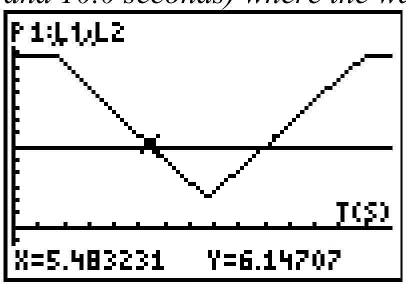
1. **Predict and sketch the distance versus time graph of the volunteer's walk in the space below.**
   Sample sketches may vary. Teachers may have different predictions about the walk.

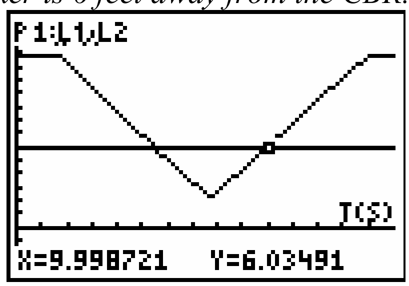
2. **Using the Ranger program on the APPS menu of the calculator, a CBR and a link cable, collect data on the relay race. You may want to move to an area that provides room for you to walk.**

3. **What is the shape of the graph of your walk? How does the graph of the walk compare to your prediction?**
   The walk is in the shape of a "v." A sample walk is shown below. Responses may vary.

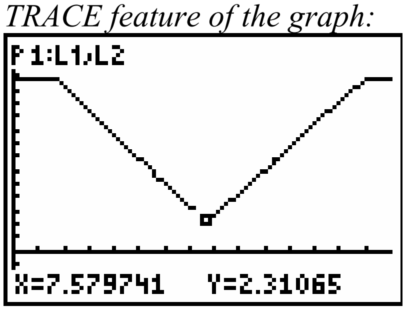


4. **How many times is the walker a given distance from the CBR?**
   At two different times the walker is the same distance from the CBR. The walker is at a given distance from the CBR one time during the walk toward the CBR and a second time on the walk away from the CBR. For example, for the sample data, there are two times (5.5 seconds and 10.0 seconds) where the walker is 6 feet away from the CBR.

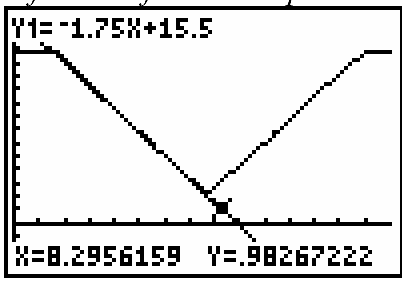




5. **At what point on the graph does the direction of the walk change? How can you interpret this point in terms of the time and distance?**
   Responses may vary. In the sample graph, the point is (7.57, 2.31). At about 7.57 seconds, the walker was about 2.31 feet from the CBR when he/she changed direction. Using the TRACE feature of the graph:

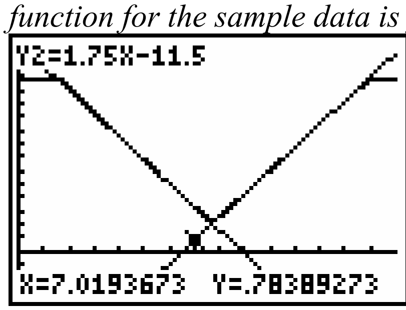


6. What part of the graph represents your motion toward the CBR? What function rule best describes the walk toward the CBR?
Responses may vary. The first part of the graph (from when the walker began the walk to the point that he/she reversed direction) represents the motion toward the CBR.
A function for the sample data is $y = -1.75x + 15.5$.



7. What is the domain for this part of the walk? How does this domain compare to the domain of the function?
Responses may vary. The domain for the sample data is $1.77 \le x \le 7.57$ *. The domain for the function is the set of all real numbers. The domain for the first part of the walk is a subset of the domain of the function.*

8. What part of the graph represents motion away from the CBR? What function rule best describes the walk away from the CBR?
Responses may vary. The second part of the graph (from when the walker reversed direction to the point that the CBR stopped collecting data) represents motion away from the CBR. A function for the sample data is $y = 1.75x - 11.5$.



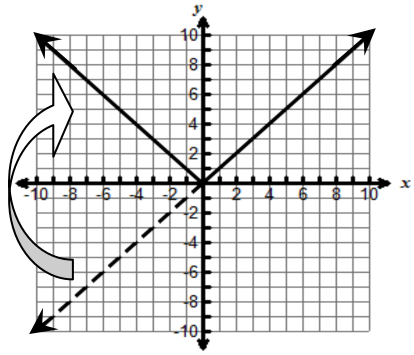
9. What is the domain for this part of the walk? How does this domain compare to the domain of the function?
Responses may vary. The domain for the sample data is $7.57 \le x \le 12.92$ *. The domain for the function is the set of all real numbers. The domain for the second part of the walk is a subset of the domain of the function.*

10. How do the functions compare? Does this match your expectation? If there are differences, what might explain them?
Responses may vary. The equations in the sample data are not "opposites" of each other. The rate walking toward the CBR, –1.75 feet per second, is the additive inverse, or "opposite" of the rate walking away from the CBR, 1.75 feet per second.

11. How can we write one function rule that describes the entire walk?
We can write an absolute value function to describe the entire walk.

12. How does this function remedy domain restrictions we encountered by using two linear functions?

The absolute value function does not require domain restrictions, because the definition of absolute value produces a vertical reflection of the graph for x-values less than 7.57.



13. What is the parent function for absolute value?

The parent function is $f(x) = |x|$.

14. What are the characteristics of absolute value functions?

Absolute value functions graph in the shape of a "v." The slope of the left side of the graph is the opposite of the slope of the right side of the graph. The graph has line symmetry through a point (the vertex).The absolute value parent function is produced by reflecting across the x-axis the part of the graph of y = x to the left of x = 0.

15. How can we use what we know about the linear functions we wrote to write one absolute value function that fits the graph of the walk? Explain your answer.

Responses may vary. We can rewrite the function $y = -1.75x + 15.5$ *as* $y = 1.75(-x) + 15.5$. *By combining that function with the function* $y = 1.75x - 11.5$, *we can write the absolute value function* $y = 1.75|x|$. *The point on the graph,(7.57, 2.31),where the direction of the walk changes represents the vertex of the absolute value function. The function for the sample is* $y = 1.75|x - 7.57| + 2.31$.

16. Write a summary statement about how modeling an absolute value function through an activity such as The Relay Race connects real-life situations to Algebra II concepts.

Responses may vary. Modeling an absolute value function allows students to connect motion and time versus distance graphs to Algebra II functions. Through modeling, students make connections among physical, graphical, and symbolic representations.

<u>Part 3</u>: Transformations of the Parent Function

Leader notes:
In this part of the professional development, participants investigate transformations of the absolute value parent function through parameter changes of *a, h,* and *k*. Ask teachers to complete the tables in this section. One of the multiple representations of an absolute value function is given in each row of the table. You may need to assist participants who have questions. At the end of this part, participants should be able to connect the symbolic, graphical, tabular, and verbal representations of absolute value functions and relate the given function to the parent function.

Use the following facilitation questions as needed to assist teachers in completing the multiple representations of **Transformations on the Parent Function**.

Facilitation Questions:

- If you are given an equation, how can you use the graphing calculator to assist you in sketching the graph or completing the table?
  You can write the equation in y= and use the graph and table features of the calculator.

- If you are given a graph, how can you use the graphing calculator to assist you in completing the table or finding the equation?
  You can use the STAT feature to plot the points and then find the equation through trial and error. You can also compare the coordinates of the data points to the coordinates of the parent function to predict the changes to the parent function and the resulting change to the parent function.

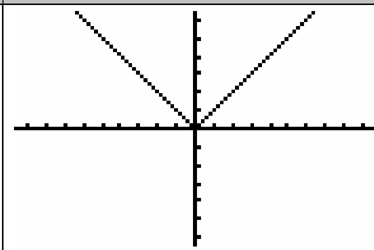
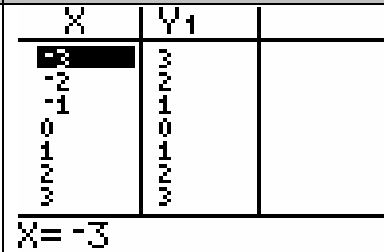
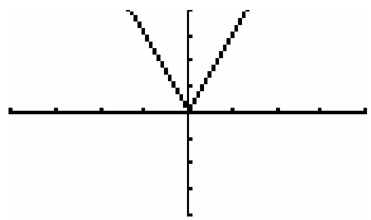
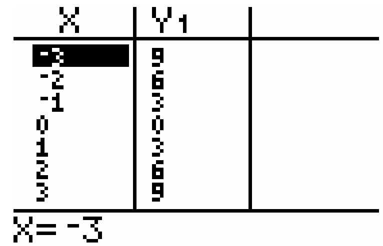
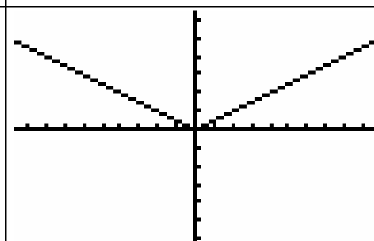
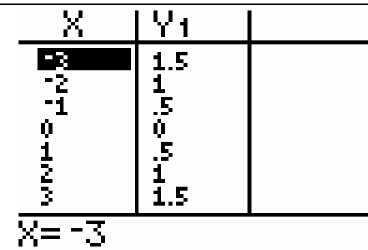
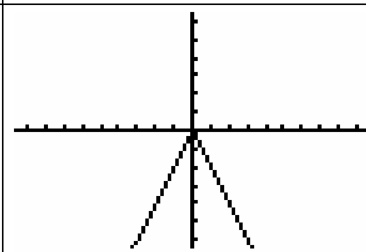
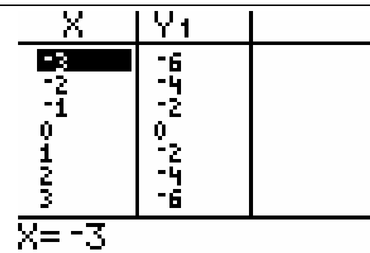
- If you are given a table, how can you use the graphing calculator to assist you in sketching the graph or finding the equation?
  You can use the STAT feature to plot the points and then find the equation through trial and error. You can also compare the coordinates of the data points to the coordinates of the parent function to predict the changes to the parent function and the resulting change to the parent function.

- How can you write a verbal description of the effect on the parent function?
  You can relate the resulting graph to the graph of the parent function in terms of vertical stretches and compressions, reflections, and horizontal and vertical shifts.

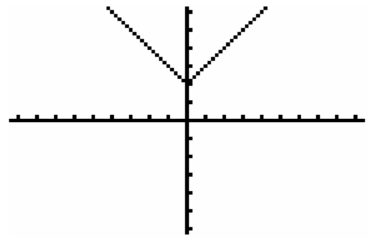
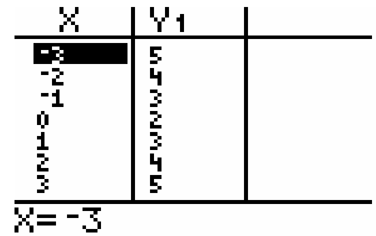
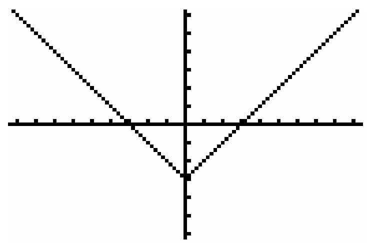
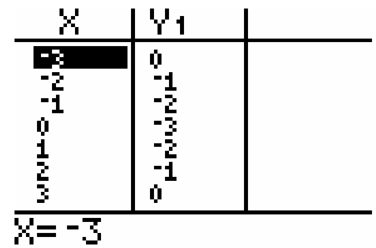
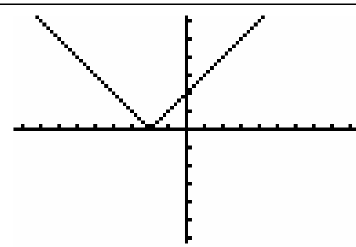
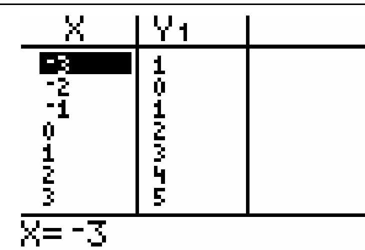
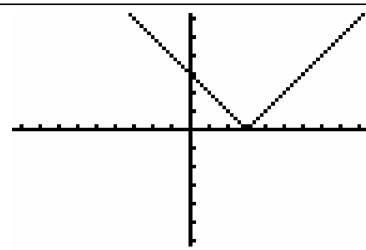
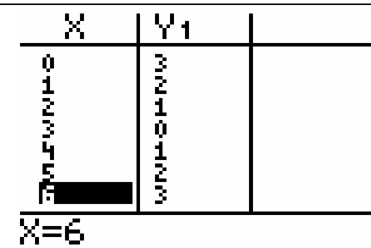
- If you are given the verbal description of the effect on the parent function, how can you use what you know about transformations to assist you in writing the equation, sketching the graph, or making a table?
  You can use what you know about transformations to write an equation and compare its graph to the graph of the parent function.

- What window can you use to get a graph that is similar to the one in the table?
  (–9.4, 9.4, 1, –6.2, 6.2, 1,1)

Transformations of the Parent Function: Changes to *a*

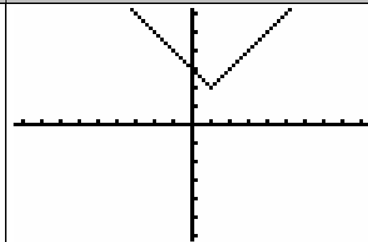
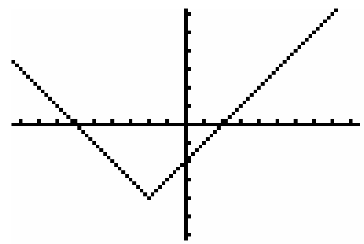
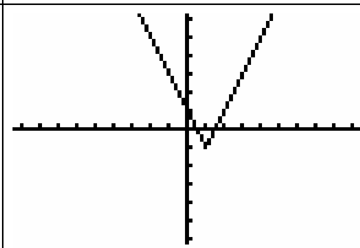
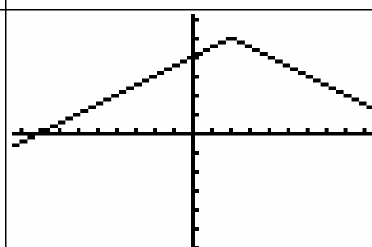
<table>
<tr><th>Function</th><th>Graph</th><th>Table</th><th>Effect on Parent Function</th></tr>
<tr><td>$y = |x|$</td><td></td><td>
</td><td>There is no effect on the parent function.</td></tr>
<tr><td>$y = 3|x|$</td><td></td><td>
</td><td>The y-values of the parent function are multiplied by a factor of 3.
(There is a vertical stretch in the parent function by a factor of 3.)</td></tr>
<tr><td>$y = \frac{1}{2}|x|$</td><td></td><td>
</td><td>The y-values of the parent function are multiplied (vertically compressed) by a factor of $\frac{1}{2}$.</td></tr>
<tr><td>$y = -2|x|$</td><td></td><td>
</td><td>The y-values of the parent function have been multiplied by a factor of –2, which vertically stretches the graph of the parent function and reflects it across the x-axis.</td></tr>
<tr><td colspan="4">Write a generalization about how the changes in the parameter a affect the graph of the parent function.
The parameter a vertically stretches or compresses the graph of the parent function. If a is positive, the graph has the same orientation as the parent function. If a is negative, the graph is reflected across the x-axis.</td></tr>
</table>

Transformations of the Parent Function: Changes to *h* and *k*

Function	Graph	Table	Effect on Parent Function
$y = \lvert x \rvert + 2$			*The y-coordinate of the vertex is increased by 2 units.* *The graph of the parent function is shifted vertically by 2 units.*
$y = \lvert x \rvert - 3$			*The y-coordinate of the vertex is decreased by 3 units.* *The graph of the parent function is shifted vertically by –3 units.*
$y = \lvert x + 2 \rvert$			The *x*-coordinate of the vertex is moved to the solution for $x + 2 = 0$. The graph of the parent function is shifted to the left by 2 units.
$y = \lvert x - 3 \rvert$			*The x-coordinate of the vertex is moved to the solution for $x - 3 = 0$.* *The graph of the parent function is shifted to the right by 3 units.*

Write a generalization about how changes in the parameters *h* and *k* affect the graph of the parent function.
The parameter k shifts the parent function vertically such that k is the y-coordinate of the vertex. The parameter h shifts the parent function horizontally so that h is the x-coordinate of the vertex.

Transformations of the Parent Function: Changes to a, *h*, and *k*

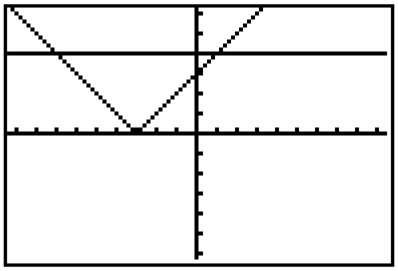
Function	Graph	Table	Effect on Parent Function
$y=\lvert x-1\rvert+2$		X: -3, -2, -1, 0, 1, 2, 3 Y1: 6, 5, 4, 3, 2, 3, 4 X= -3	*The vertex of the parent function has been translated to (1, 2).*
$y=\lvert x+2\rvert-4$		X: -3, -2, -1, 0, 1, 2, 3 Y1: -3, -4, -3, -2, -1, 0, 1 X= -3	The vertex of the parent function has been translated to (–2, –4).
$y=2\lvert x-1\rvert-1$		X: -3, -2, -1, 0, 1, 2, 3 Y1: 7, 5, 3, 1, -1, 1, 3 X= -3	*The parent function has been stretched vertically by a factor of 2 and the vertex has been translated to (1,–1).*
$y=-\frac{1}{2}\lvert x-2\rvert+5$		X: -3, -2, -1, 0, 1, 2, 3 Y1: 2.5, 3, 3.5, 4, 4.5, 5, 4.5 X= -3	*The parent function has been compressed vertically by a factor of* $-\frac{1}{2}$ *and the vertex has been translated to (2, 5).*

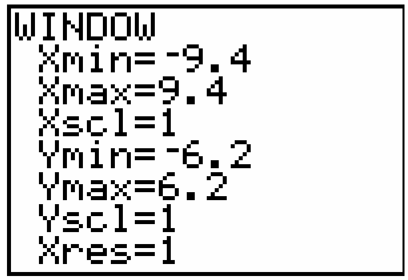
Write a generalization about how changes in the parameters *a, h, and k* affect the graph of the parent function.
The parameter a vertically stretches or compresses the graph of the parent function by scaling the y-values by a factor of a. The vertex of the parent function is shifted to the point (h,k).

<u>Part 4</u>: Solving Absolute Value Equations and Inequalities

Leader Notes:
In this part of the professional development participants investigate connections among the algebraic and graphical solutions of absolute value equations and inequalities.

1. **Consider the system of equations $y = |x+3|$ and $y = 4$. Graph the system and sketch the graph. Describe your viewing window.**



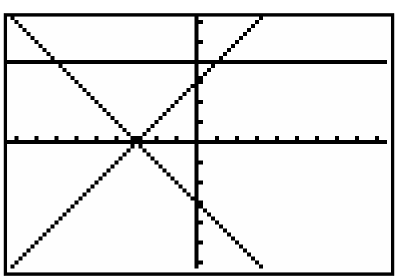


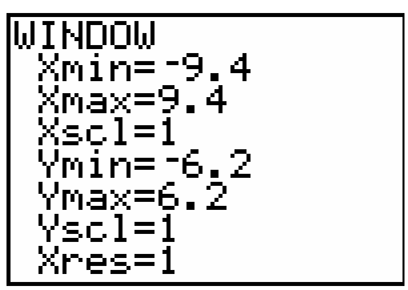
   a) **What are the domain and range for each function in this system?**
   For the function, $y = |x+3|$, the domain is the set of all real numbers and the range is the set of all y-values greater than or equal to 0. For the function, y = 4 ,the domain and is the set of all real numbers, and the range is the set that contains 4.

   b) **What are the coordinates of the points that are solutions for this system? Why are there two solutions?**
   The solutions are (–7, 4) and (1, 4). There are two solutions because the line $y = 4$ intersects $y = |x+3|$ in two points.

   c) **How can you use the concept of substitution to write this system as one equation?**
   $|x+3| = 4$

2. **Graph the functions $y = x+3$, $y = -(x+3)$ and $y = 4$ and sketch the graph. Describe your viewing window.**



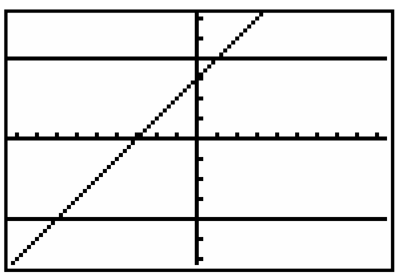


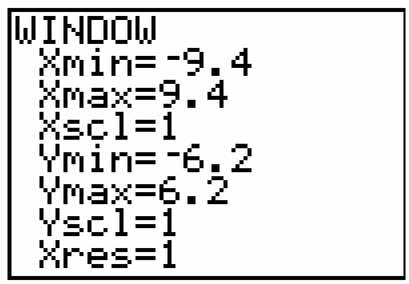
a) What are the domain and range for each function in this system? How does this system of equations compare to the original system?
The domain and range for $y = x + 3$ and $y = -(x + 3)$ is the set of all real numbers. The domain of $y = 4$ is the set of all real numbers and the range is the set that contains 4. The solution for this system is the same as the original system. The absolute value function is represented by parts of the lines $y = x + 3$ and $y = -(x + 3)$.

b) What are the coordinates of the points that are solutions for this system? How do these solutions compare to the solutions of the original system?
The solutions for this system are (–7, 4) and (1, 4). The solution for this system is the same as the original system.

c) How can you use the concept of substitution to write this system of three equations as a system of two equations?
$x + 3 = 4$ *and* $-(x + 3) = 4$

3. Graph the functions $y = x + 3$, $y = 4$, and $y = -4$, and sketch the graph. Describe your viewing window.





a) What are the domain and range for each function in this system?
The domain and range of $y = x + 3$ is the set of all real numbers. The domain of $y = 4$ and $y = -4$ is the set of all real numbers. The range of $y = 4$ is the set that contains 4. The range of $y = -4$ is the set that contains –4.

b) What are the coordinates of the points that are solutions for this system? How do the solutions for the graph above compare to the original solutions in question 1?
The solutions for this system are (–7, –4) and (1, 4). Only one solution is the same as the original system, (1, 4).

c) Compare the graph above and the graph in question 2 to the graph of the original system. Which graph is conceptually related to the graph of the original system? Which graph is not conceptually related? Why?
The graph in question 2 is conceptually related to the graph of the original system. The graph above is not. The system of equations in question 2 represents the characteristics of the absolute value function and the horizontal line $y = 4$. The system above has two horizontal lines, one of which does not represent the system. Also, the equation $y = x + 3$ only represents half of the absolute value function.

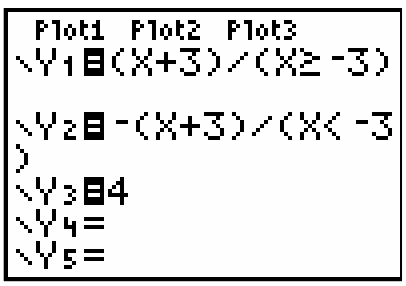
d) What misconceptions might arise by setting up a process to solve $x+3=4$ and $x+3=-4$?

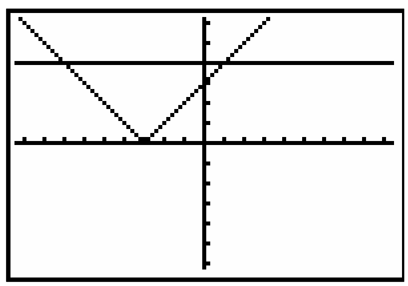
This system is conceptually and graphically different from the original problem and does not represent the intersection of the absolute value function and the line y = 4. Symbolically, the x-coordinates of the solutions are the same, but the y-coordinates are different. So the ordered pairs for this solution do not match the ordered pairs for the original system.

e) What restrictions do we need to place on the domains of the functions $y=x+3$ and $y=-(x+3)$ so that their graphs match the graph of the function, $y=|x+3|$?

The domain of the function $y=x+3$ should be restricted to $x \geq -3$; the domain of the function $y=-(x+3)$ should be restricted to $x<-3$.

f) Graph the system of equations with the restrictions. Sketch your graph.





g) How does the graph of this system of equations and its solutions compare to the graph and solutions to the original system in question 1?

The graphs and solutions are the same.

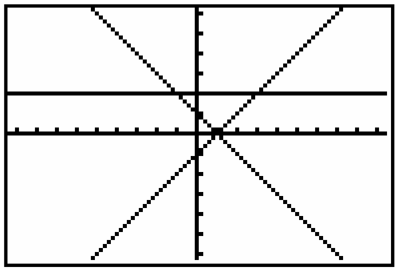
4. Write a system of three equations that conceptually relate to the system of equations $|x-1|=2$.

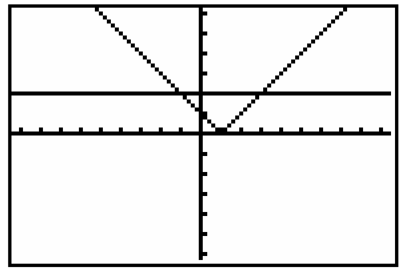
$y=x-1$

$y=-(x-1)$

$y=2$

a) Graph the system you wrote and compare it to the graph of equations $y = |x-1|$ and $y = 2$. How do the graphs compare? How do the solutions compare?





Parts of the graphs of the linear equations in my system represent the graph of the absolute value function. The solutions are the same.

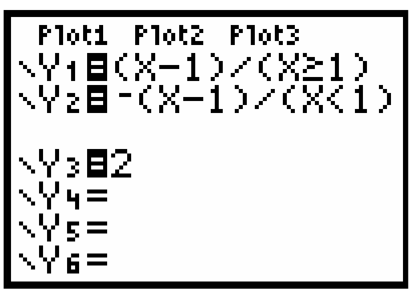
b) How do the equations $x - 1 = 2$ and $-(x-1) = 2$ relate to the graphs of the above systems?

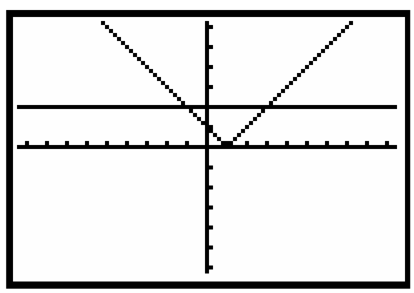
The equations represent the intersection of the absolute value function and the line $y = 2$.

c) What restrictions should we place on the domains of the functions in your system so that the graph of your system matches the graph of $y = |x-1|$?

The domain of the function $y = x - 1$ should be restricted to $x \ge 1$. The domain of the function $y = -(x-1)$ should be restricted to $x < 1$.

d) Graph the system of equations with the restrictions. Sketch your graph.





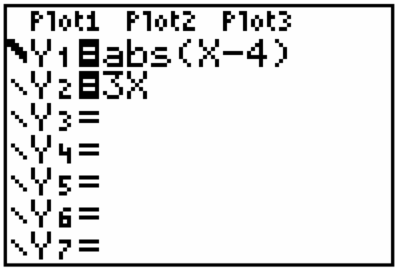
e) How does the graph of this system of equations and its solutions compare to the graph and solutions of $y = |x-1|$ and $y = 2$ in question 4a?

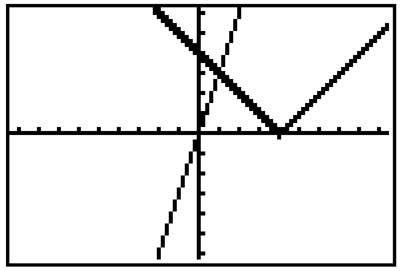
The graphs and solutions are the same.

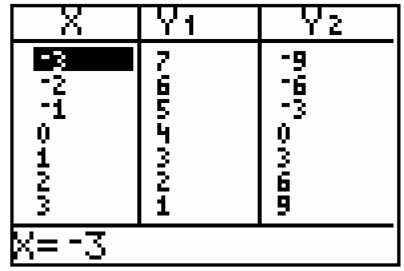
5. Write a statement about the connection between an equation such as $|x+2| = 5$ and the system $x + 2 = 5$ and $-(x+2) = 5$.

Responses may vary. The system $x + 2 = 5$ and $-(x+2) = 5$ are conceptually the same as $|x+2| = 5$ and have the same solution set.

6. Consider the system of equations represented by the equation $|x-4| = 3x$. Graph the equations $y = |x-4|$ and $y = 3x$, sketch the graph, and complete the table. (Hint: You may want to bold the absolute value equation.)



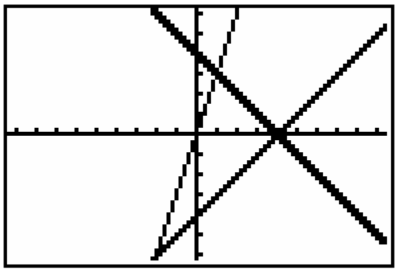
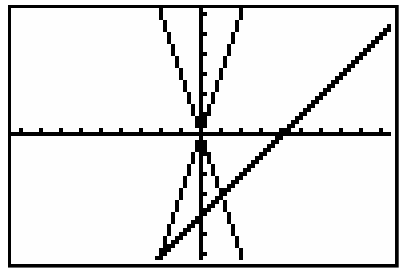




a) How many solutions does this system have? Why?

This system has one solution (1, 3) because the line $y = 3x$ intersects $y = |x-4|$ in only one point.

b) Compare the following systems in the table graphically, tabularly, and algebraically. (Hint: you may want to bold the equation(s) representing the absolute value function.)

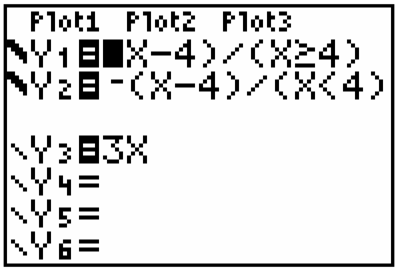
<table>
<tr><th>System A</th><th>System B</th></tr>
<tr><td>$y = x - 4$
$y = -(x-4)$
$y = 3x$</td><td>$y = x - 4$
$y = 3x$
$y = -3x$</td></tr>
<tr><td></td><td></td></tr>
<tr><td>
<table>
<tr><th>x</th><th>y_1</th><th>y_2</th><th>y_3</th></tr>
<tr><td>–3</td><td>–7</td><td>7</td><td>–9</td></tr>
<tr><td>–2</td><td>–6</td><td>6</td><td>–6</td></tr>
<tr><td>–1</td><td>–5</td><td>5</td><td>–3</td></tr>
<tr><td>0</td><td>–4</td><td>4</td><td>0</td></tr>
<tr><td>1</td><td>–3</td><td>3</td><td>3</td></tr>
<tr><td>2</td><td>–2</td><td>2</td><td>6</td></tr>
<tr><td>3</td><td>–1</td><td>1</td><td>9</td></tr>
</table>
</td><td>
<table>
<tr><th>x</th><th>y_1</th><th>y_2</th><th>y_3</th></tr>
<tr><td>–3</td><td>–7</td><td>–9</td><td>9</td></tr>
<tr><td>–2</td><td>–6</td><td>–6</td><td>6</td></tr>
<tr><td>–1</td><td>–5</td><td>–3</td><td>3</td></tr>
<tr><td>0</td><td>–4</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>–3</td><td>3</td><td>–3</td></tr>
<tr><td>2</td><td>–2</td><td>6</td><td>–6</td></tr>
<tr><td>3</td><td>–1</td><td>9</td><td>–9</td></tr>
</table>
</td></tr>
<tr><td>$-(x-4) = 3x$ *or* $x - 4 = 3x$
$-x + 4 = 3x$ $-2x = 4$
$-x = -4$ $x = -2$
$x = 1$ $y = -6$
$y = 3$ $(-2,-6)$
$(1,3)$</td><td>$x - 4 = -3x$ *or* $x - 4 = 3x$
$4x = 4$ $-2x = 4$
$x = 1$ $x = -2$
$x = -3$ $y = -6$
$y = 3$ $(-2,-6)$
$(1,-3)$</td></tr>
</table>

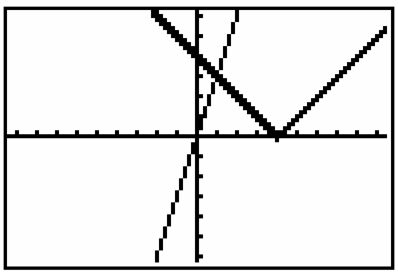
c) What are the solutions for System A? What are the solutions for System B? How do those solutions compare to the original system?

The solutions for System A are (1, 3) and (–2, –6). The solutions for System B are (1, –3) and (–2, –6). Only System A has a solution that is the same solution as the original system.

d) **Which system, A or B, conceptually relates to the original system?**
System A relates conceptually to the original system.

e) **Restrict the domains for the functions in System A, graph the new system, and complete the table. Find the table values for the solutions.**





x	y_1	y_2	y_3
–3	*ER*	*7*	*–9*
–2	*ER*	*6*	*–6*
–1	*ER*	*5*	*–3*
0	*ER*	*4*	*0*
1	*ER*	*3*	*3*
2	*ER*	*2*	*6*
3	*ER*	*1*	*9*

f) **How are the solutions for System A without the restricted domain and System A with the restricted domain alike and different?**
Both systems have the solution (1, 3). System A with an unrestricted domain has as extra solution (–2, –6).

g) **Why does System A without the restricted domain produce two solutions? Which solution of A is not a solution of the original system? What do we call that solution?**
System A produces two solutions through the algebraic process. The solution (–2, –6) is not a solution of the original system. We call this solution an extraneous root.

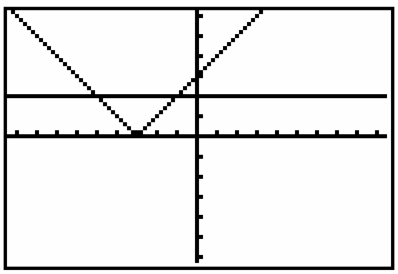
7. **What understanding does graphing a system involving absolute value equations provide with regard to the actual number of solutions to the system and the corresponding equations that intersect? How does the graphical solution connect to the algebraic process?**
By graphing the system, we can see how many roots the system has. We can also see where the solution is located and the corresponding equations that intersect. By graphing the system first, we can see which equations intersect to form the solutions. We can also avoid extraneous roots.

8. Write a statement comparing the common algebraic process for solving absolute value equations to the conceptual understanding of the solutions to a system of absolute value equations.

The common algebraic process of setting the principal part of the absolute value equal to the positive and negative right-hand side quantities does not align conceptually to the graphical solution of the equation and may give us the same x-values, but not the same y-values. We may also be solving an equation where a root does not exist in the domain (an extraneous root).

9. Consider the system that represents the inequality $|x+3| > 2$.

a) Show the solution graphically, tabularly, and symbolically.



X	Y1	Y2
-6	3	2
-5	2	2
-4	1	2
-3	0	2
-2	1	2
-1	2	2
0	3	2

X= -6

$x \geq -1$ *or* $x \leq -5$

b) When we ask students to show us where $|x+3| > 2$, what are we asking?

We are asking students to find the x-values where the corresponding y-values for the function $y = |x+3|$ *are greater than the y-values of* $y = 2$.

Explain

Leaders' Note: The Maximizing Algebra II Performance (MAP) professional development is intended to be an extension of the ideas introduced in Mathematics TEKS Connections (MTC). Throughout the professional development experience, we allude to components of MTC such as the Processing Framework Model, the emphasis of making connections among representations, and the links between conceptual understanding and procedural fluency.

Debriefing the Experience:

1. **What concepts did we explore in the previous set of activities? How were they connected?**
   Responses may vary. Participants should observe that investigating the physical and graphical representations of absolute value functions enhances understanding of the concept of absolute value. Absolute value equations and inequalities can be solved by graphing the equations and inequalities as systems.

2. **What procedures did we use to describe absolute value functions? How are they related?**
   Tabular, graphical, and symbolic procedures were all used throughout the Explore phase. Ultimately, they are all connected through the numerical relationships used to generate them.

3. **What knowledge from Algebra I do students bring about absolute value functions?**
   Absolute value is not a student expectation until Algebra II, so students may not have had prior instruction on the absolute value of a number or absolute value functions.

4. **After working with absolute value functions in Algebra II, what are students' next steps in Precalculus or other higher mathematics courses?**
   According to the Precalculus TEKS, students will expand their understanding of absolute value functions. They will be expected to apply basic transformations and compositions with absolute value functions, including $\left|f\left(x\right)\right|$, *and* $f\left(\left|x\right|\right)$, *to the parent functions.*

Anchoring the Experience:

5. **Distribute to each table group a poster-size copy of the Processing Framework Model.**

6. **Ask each group to respond to the question:**

   Where in the processing framework would you locate the different activities from the Explore phase?

7. **Participants can use one color of sticky notes to record their responses. In future Explain phases, participants will use other colors of sticky notes to record their responses.**

Horizontal Connections within the TEKS

8. Direct the participants' attention to the second layer in the Processing Framework Model: Horizontal Connections among Strands.

9. Prompt the participants to study the Algebra II TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.

10. Invite each table group to share 2 connections that they found and record them so that they are visible to the entire group.

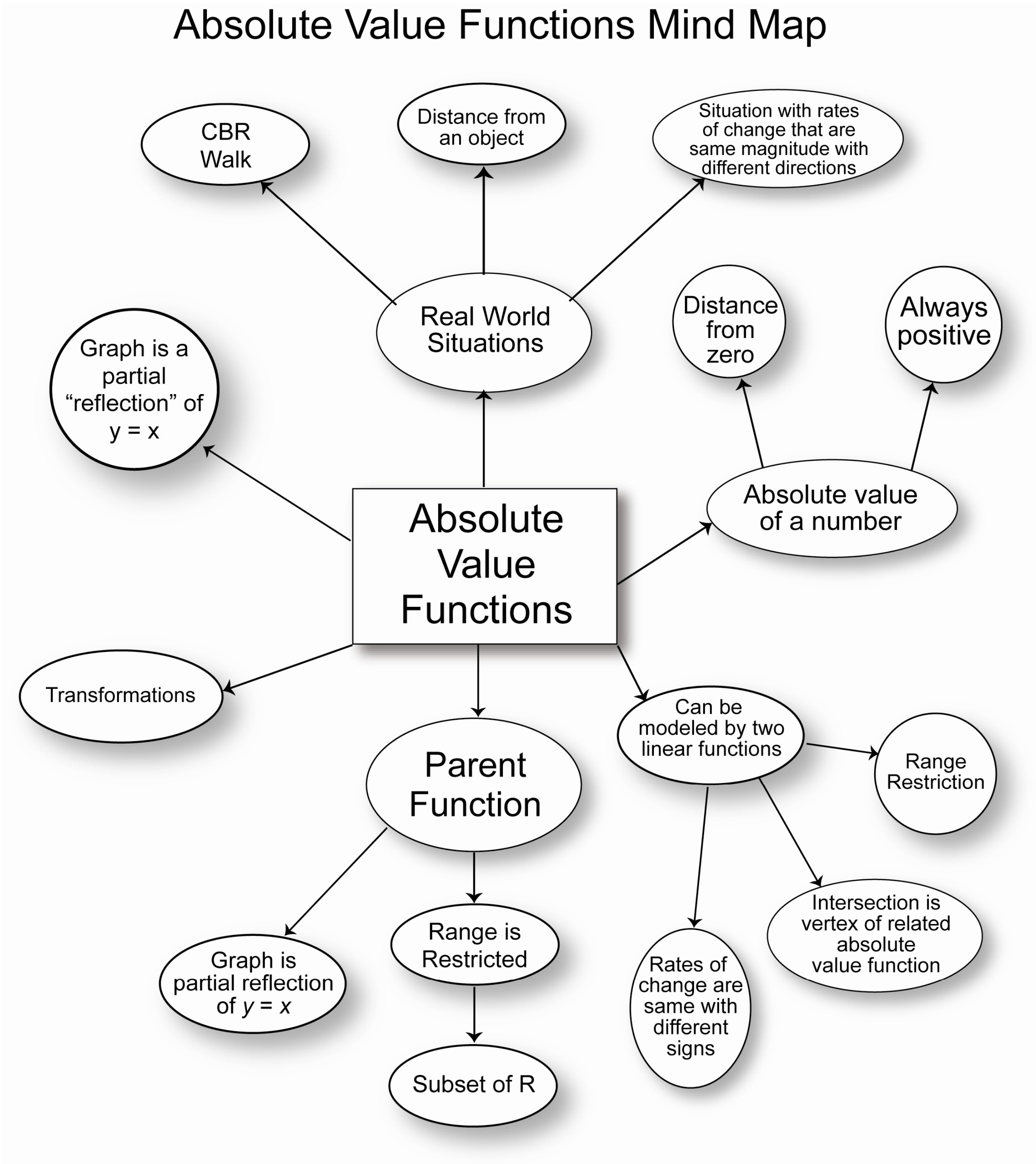
Vertical Connections within the TEKS

11. Direct the participants' attention to the third layer in the Processing Framework Model: Vertical Connections across Grade Levels.

12. Prompt the participants to study the Algebra I, Geometry, Math Models, and Precalculus TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.

13. Invite each table group to share 2 connections that they found, recording so that the entire large group may see.

14. Provide each group of participants with a clean sheet of chart paper. Ask them to create a "mind map" for the mathematical term of "absolute value."



15. **Provide an opportunity for each group to share the mind maps with the larger group. Discuss similarities, differences, and key points brought forth by participants.**

16. **Distribute the vocabulary organizer template to each participant. Ask participants to construct a vocabulary model for the term "absolute value functions."**

17. When participants have completed their vocabulary models, ask participants to identify strategies from their experiences so far in the professional development that could be used to support students who typically struggle with Algebra II topics.

Note to Leader: *You may wish to have each small group brainstorm a few ideas first, then share their ideas with the large group while you record their responses on a transparency or chart paper.*

18. How would this lesson maximize student performance in Algebra II for teaching and learning the mathematical concepts and procedures associated with absolute value functions?
Responses may vary. Anchoring procedures within a conceptual framework helps students understand what they are doing so that they become more fluent with the procedures required to accomplish their tasks. Problems present themselves in a variety of representations; providing students with multiple procedures to solve a given problem empowers students to solve the problem more easily.

Elaborate

Leaders' Note: *In this phase, participants will extend their learning experiences to their classroom.*

1. **Distribute the 5E Student Lesson planning template. Ask participants to think back to their experiences in the Explore phase. Pose the following task:**

   What might a student-ready 5E lesson on absolute value functions look like?
   - **What would the Engage look like?**
   - **Which experiences/activities would students explore firsthand?**
   - **How would students formalize and generalize their learning?**
   - **What would the Elaborate look like?**
   - **How would we evaluate student understanding of inverses of relations/functions?**

2. **After participants have recorded their thoughts, direct them to the student lesson for absolute value functions. Allow time for participants to review lessons.**

3. **How does this 5E lesson compare to your vision of a student-centered 5E lesson?**
   Responses may vary.

4. **How does this lesson help remove obstacles that typically keep students from being successful in Algebra II?**
   By connecting the concept of absolute value to a physical model, students gain a better understanding of the absolute value function and the limitations of using linear function to describe the model. By solving absolute value equations as systems and through graphing, solutions can be verified and connected to the algebraic processes commonly used to find those solutions. Students can use alternate methods with which to solve meaningful problems.

5. **How does this lesson maximize your instructional time and effort in teaching Algebra II?**
   Taking time to create a solid conceptual foundation reduces the need for re-teaching time and effort and increases student participation in the learning process. Conceptual connections to algebraic process strengthen the understanding of absolute value functions and mirror the links among multiple representations.

6. **How does this lesson maximize student learning in Algebra II?**
   Using multiple representations and foundations for functions concepts allows students to make connections among different ideas. These connections allow students to apply their learning to new situations more quickly and readily.

7. **How does this lesson accelerate student learning and increase the efficiency of learning?**
   Foundations for functions concepts such as function transformations transcend all kinds of functions. A basic toolkit for students to use when working with functions allows students to rethink what they know about linear and quadratic functions while they are learning concepts and procedures associated with other function families.

8. **Read through the suggested strategies on Strategies that Support English Language Learners. Consider the possible strategies designed to increase the achievement of English language learners.**

   As participants read through the strategies that support English language learners and strategies that support students with special needs, they may notice that eight of the ten strategies are the same. The intention is to underscore effective teaching practices for all students. However, English language learners have needs specific to language that students with special needs may or may not have. The two strategies that are unique to the English language learners reflect an emphasis on language. Students with special needs may have prescribed modifications and accommodations that address materials and feedback. Students with special needs often benefit from progress monitoring with direct feedback and adaptation of materials for structure and/or pacing. A system of quick response is an intentional plan to gather data about a student's progress to determine whether or not the modification and (or) accommodation are (is) having the desired effect. The intention of the strategies is to provide access to rigorous mathematics and support students as they learn rigorous mathematics.

9. **What evidence of these strategies do you find in this portion of the professional development?**
   Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening. This contributes to the teacher's ability to create an emotionally safe environment for learning. Tools such as the CBR and graphing calculator can be used to communicate about and solve problem situations involving absolute value functions.*

10. **Which strategies require adaptation of the materials in this portion of the professional development?**
    Responses may vary. Most of the strategies are incorporated throughout the materials.

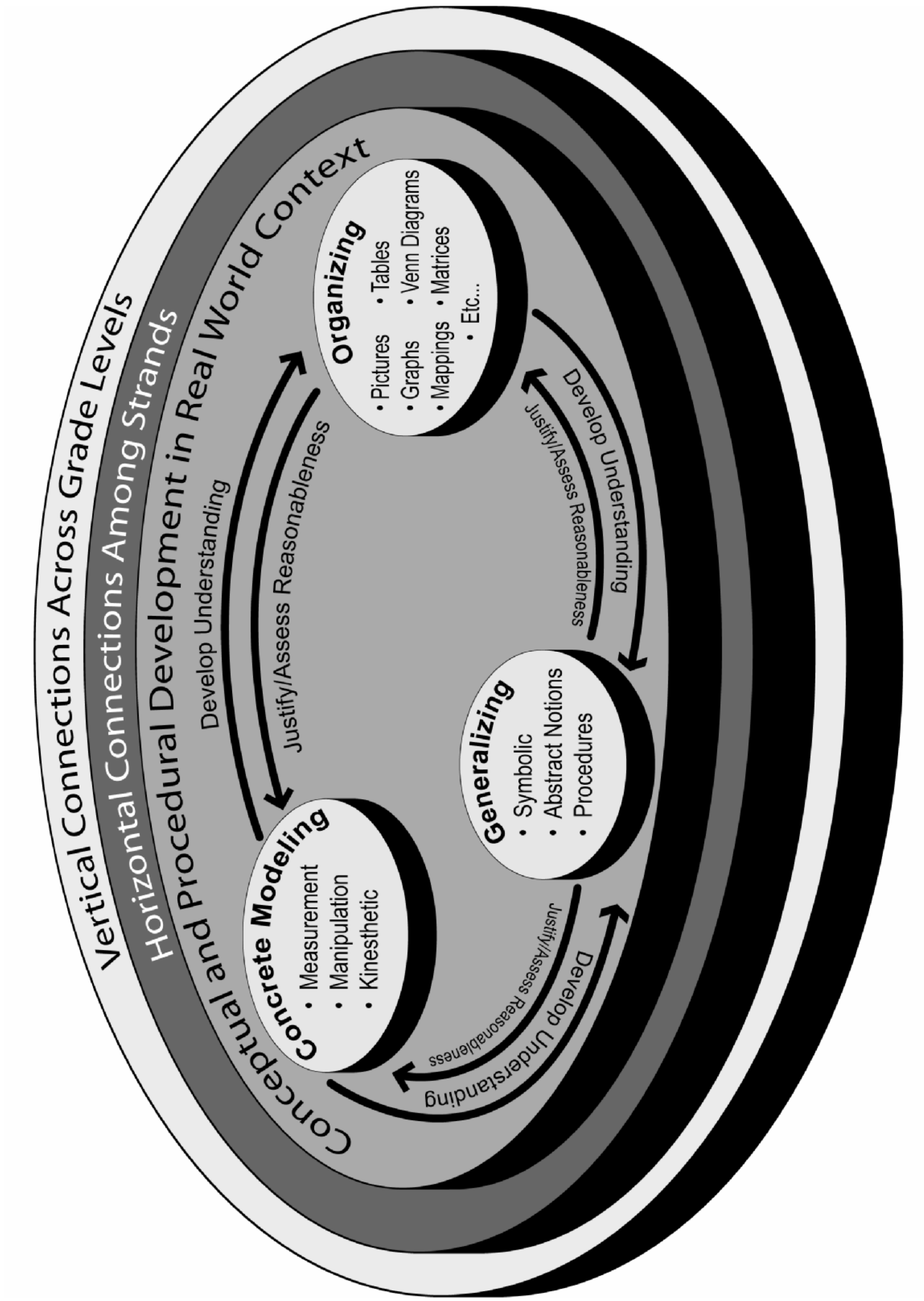
11. **Read through the suggested strategies on Strategies that Support Students with Special Needs. Consider the possible strategies designed to increase the achievement of students with special needs.**

12. **What evidence of these strategies do you find in this portion of the professional development?**
    Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening. This contributes to the teacher's ability to create an emotionally safe environment for learning. Tools such as the CBR and graphing calculator can be used to communicate about and solve problem situations involving absolute value functions.*

13. Which strategies require adaptation of the materials in this portion of the professional development?

Responses may vary. Most of the strategies are incorporated throughout the materials. Some materials may need to be modified for format.

Processing Framework Model



Vocabulary Organizer

My definition	**Personal Association**
Example	**Non-Example**

5E Lesson Planning Template

Description	Activity
Engage The activity should be designed to generate student interest in a problem situation and to make connections to prior knowledge. The instructor initiates this stage by asking meaningful questions, posing a problem to be solved, or by showing something intriguing.	
Explore The activity should provide students with an opportunity to become actively involved with the key concepts of the lesson through a guided exploration requiring them to probe, inquire, and question. The instructor actively monitors students as they interact with each other and the activity.	
Explain Students collaboratively begin to sequence events/facts from the investigation and communicate these findings to each other and the instructor. The instructor, acting in a facilitation role, formalizes student findings by providing further explanations and additional meaning or information, such as correct terminology.	
Elaborate Students extend, expand, or apply what they have learned in the first three stages and connect this knowledge with prior learning to deepen understanding. Instructors can use the Elaborate stage to verify students' understandings.	
Evaluate Evaluation occurs throughout students' learning experiences. More formal evaluation can be conducted at this stage. Instructors can determine whether the learner has reached the desired level of understanding the key ideas and concepts.	

Strategies that Support English Language Learners (ELL)

Strategy	Explore, Explain, Elaborate 3
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Be conscious of tone and diction. Speak slowly and distinctly.	
Incorporate language skills (reading, writing, speaking, and listening) into instruction.	

Strategies that Support Students with Special Needs

Strategy	Explore, Explain, Elaborate 3
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Use a system of quick response to needs and accommodations including progress monitoring to inform instruction.	
Accommodate materials for format, structure, sequence, etc. as needed.	

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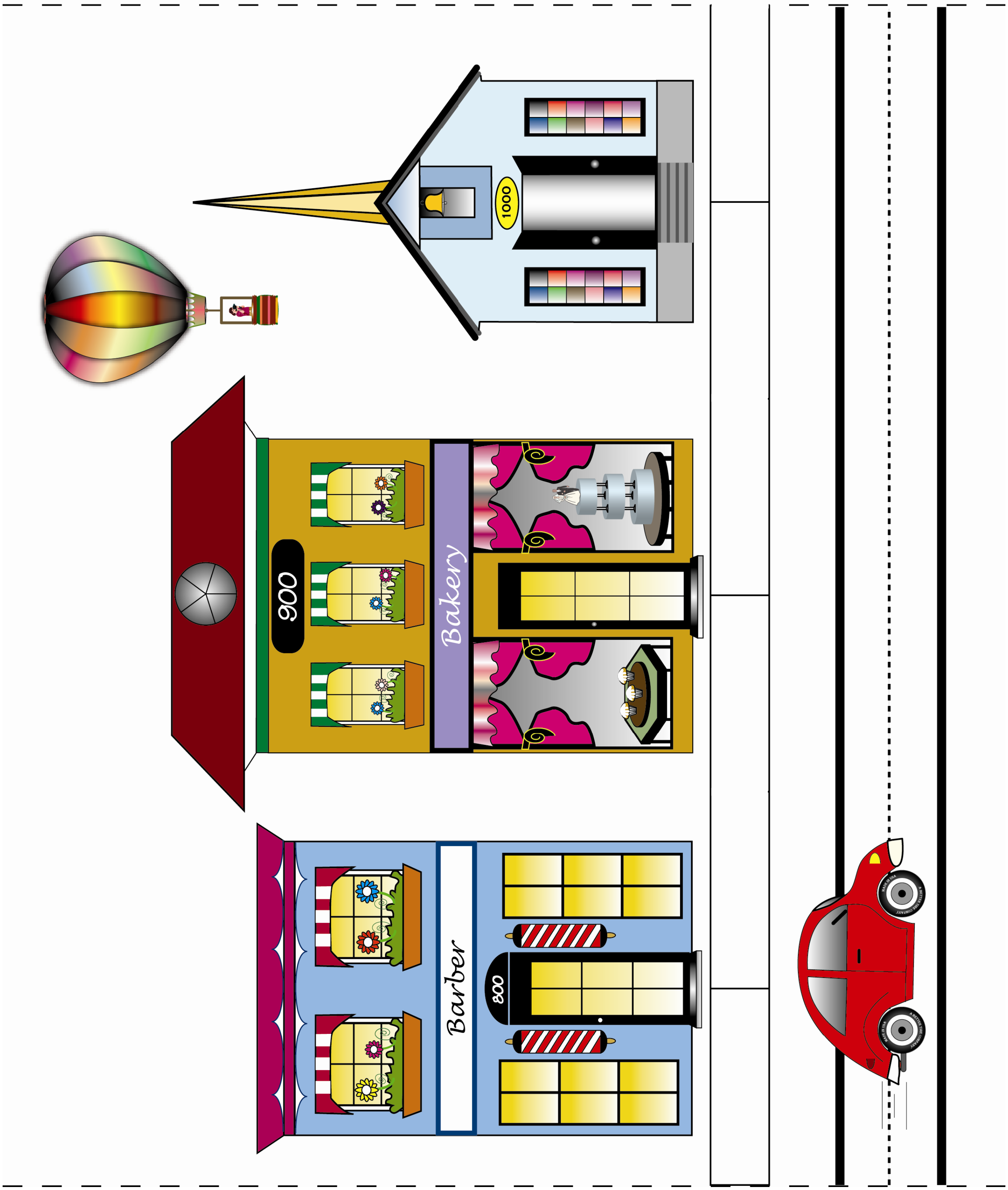

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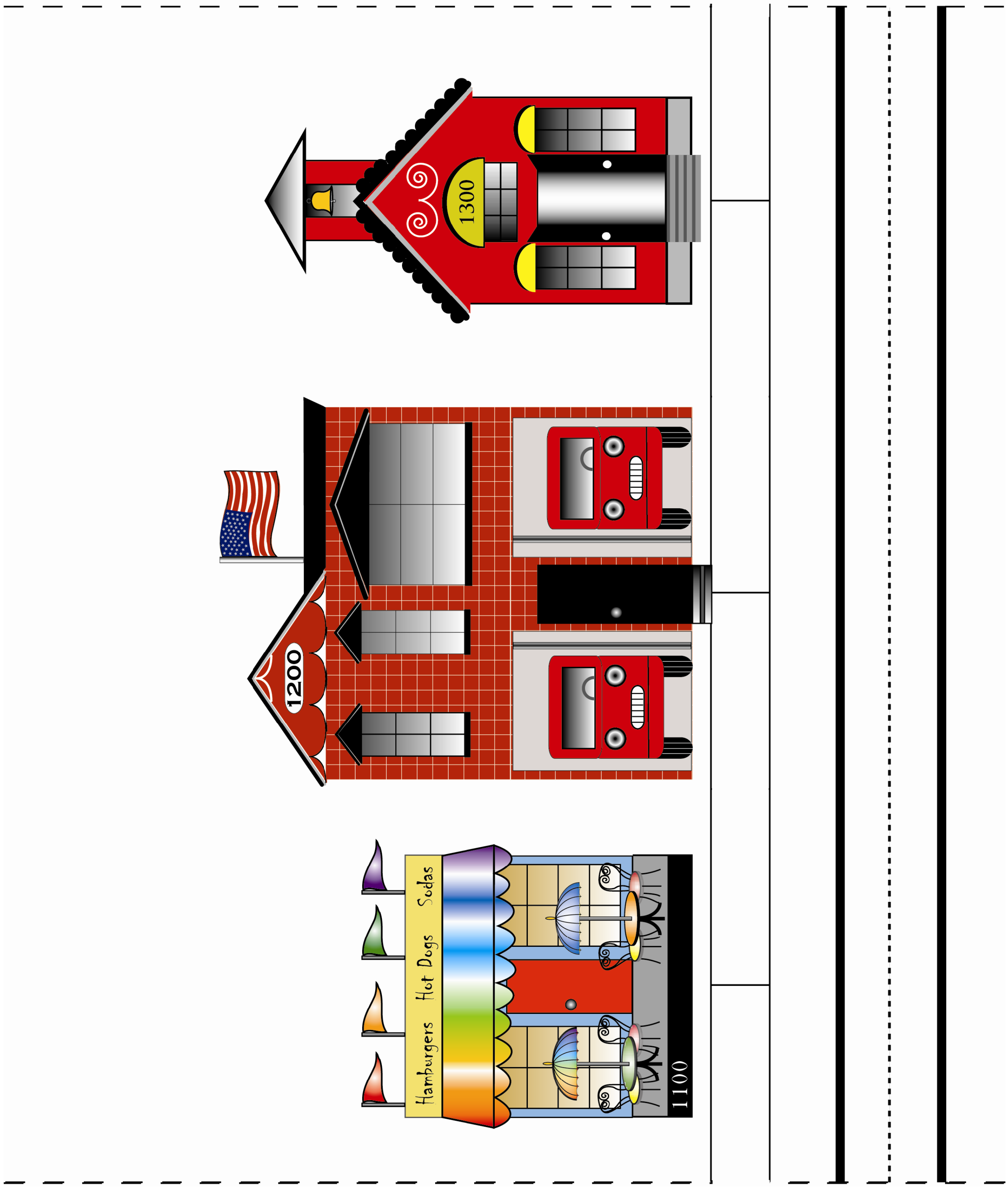
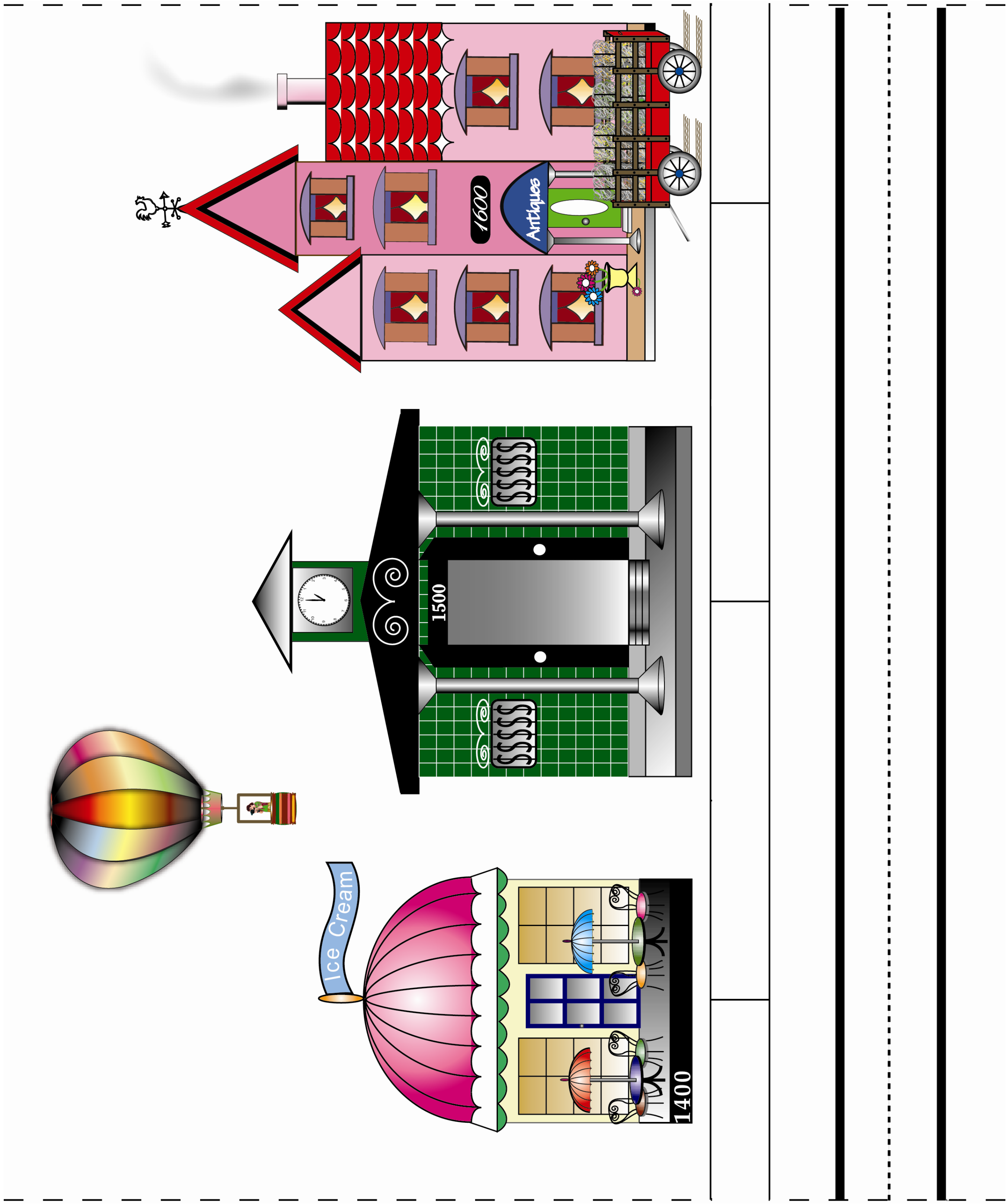



1600
Antiques
1500
Ice Cream
1400

Participant Pages: The Fire Station Problem

A fire station is located on Main Street and has buildings at every block to the right and to the left. You will investigate the relationship between the address number on a building and its distance in feet from the fire station. On average, a mile in the city is composed of 16 city blocks. So each city block is about 330 feet long (5280 feet ÷ 16 = 330 feet). Each building in centered on the block.

1. Complete the table below that relates the address of a building (x) with its distance in feet from the fire station (y).

Address Number (x)	Distance in Feet from the Fire Station (y)

2. Which building is 660 feet way from the fire station? Explain your answer.

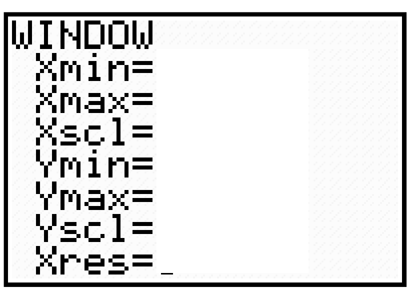
3. If we send someone to the building that is 660 feet away from the fire station, how will she know that she has arrived at the correct place?

4. What words might we use to describe two locations that are the same distance from the fire station?

5. Suppose the buildings on Main Street are renumbered as if they are on a number line so that the location of the fire station represents 0. How do we describe two numbers with the same distance from 0?

6. Draw a scatterplot that represents the data in the table.

7. Make a scatterplot of your data using your graphing calculator. Describe your viewing window.

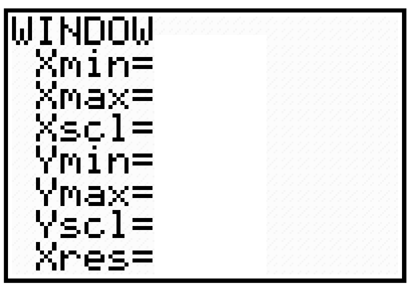


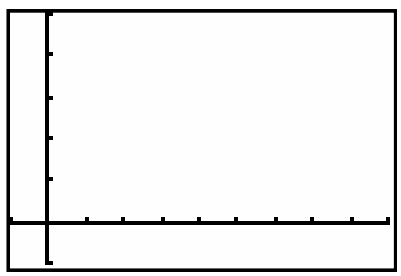
8. What function or functions might students use to describe the scatterplot?

9. Find two linear functions that pass through the data points. What process did you use to find the equations of the lines?

10. Graph the equations on your calculator. How are the equations similar? How are they different?

11. If necessary adjust the window to clearly see the intersection of the two lines. What does the intersection of these two lines represent? Sketch the graph.





12. Where do the equations fit the graph of the data points? Where do the equations not fit the graph of the data points?

13. How well do the linear equations model the data points?

14. Write summary statements about conceptual understanding of the absolute value of a number and linear equations that model a situation using absolute value.

Participant Pages: The Relay Race

Pretend you are in an unusual relay race. The object of the race is to walk toward the CBR at a slow steady rate as if to pick up something and then to walk backwards away from the CBR at the same rate without stopping. The person whose rate walking towards the CBR matches the rate walking away from the CBR and who changes direction instantly wins the race!

1. Predict and sketch the distance versus time graph of the volunteer's walk in the space below.

2. Using the Ranger program on the APPS menu of the calculator, a CBR and a link cable, collect data on the relay race. You may want to move to an area that provides room for you to walk.

3. What is the shape of the graph of your walk? How does the graph of the walk compare to your prediction?

4. How many times is the walker a given distance from the CBR?

5. At what point on the graph does the direction of the walk change? How can you interpret this point in terms of the time and distance?

6. What part of the graph represents your motion toward the CBR? What function rule best describes the walk toward the CBR?

7. What is the domain for this part of the walk? How does this domain compare to the domain of the function?

8. What part of the graph represents motion away from the CBR? What function rule best describes the walk away from the CBR?

9. What is the domain for this part of the walk? How does this domain compare to the domain of the function?

10. How do the functions compare? Does this match your expectation? If there are differences, what might explain them?

11. How can we write one function rule that describes the entire walk?

12. How does this function remedy domain restrictions we encountered by using two linear functions?

13. What is the parent function for absolute value?

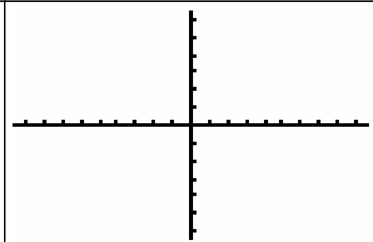
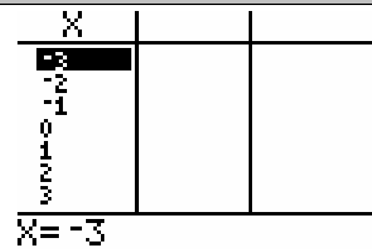
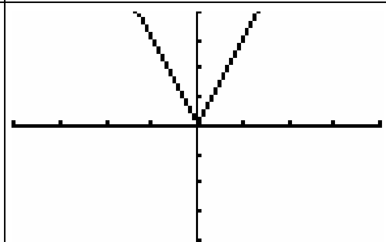
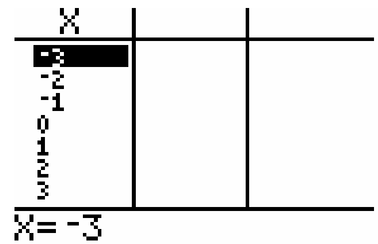
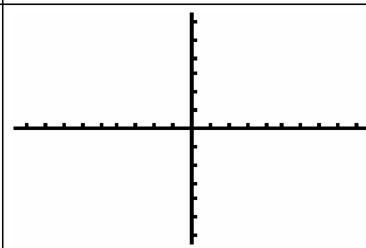
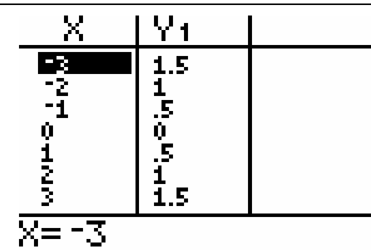
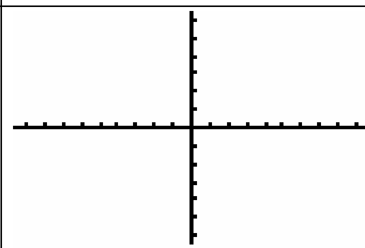
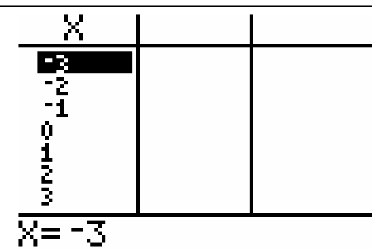
14. What are the characteristics of absolute value functions?

15. How can we use what we know about the linear functions we wrote to write one absolute value function that fits the graph of the walk? Explain your answer.

.

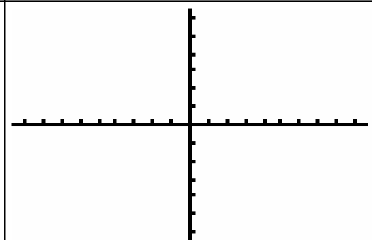
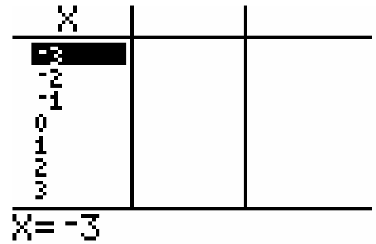
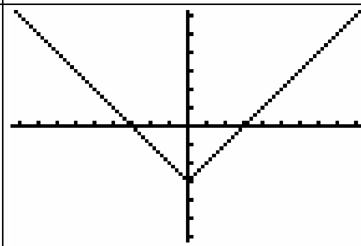
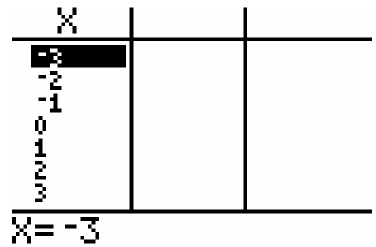
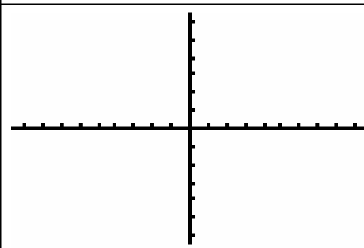
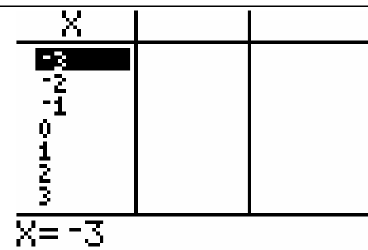
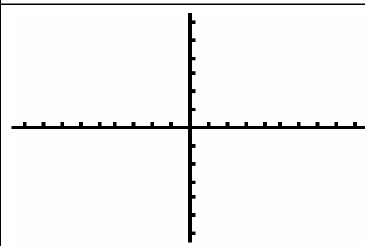
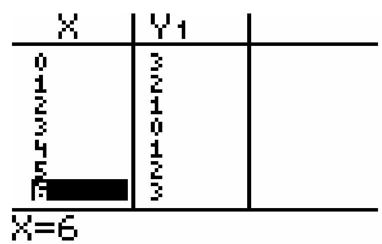
16. Write a summary statement about how modeling an absolute value function through an activity such as The Relay Race connects real-life situations to Algebra II concepts.

Participant Pages: Transformations of the Parent Function: Changes to a

Function	Graph	Table	Effect on Parent Function
$y = \|x\|$			
			
			
			The y-values of the parent function have been multiplied by a factor of –2, which vertically stretches the graph of the parent function and reflects it across the x-axis.

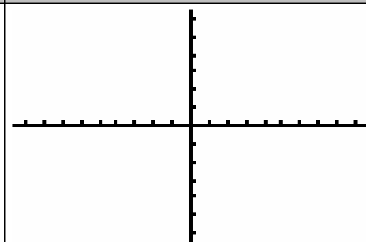
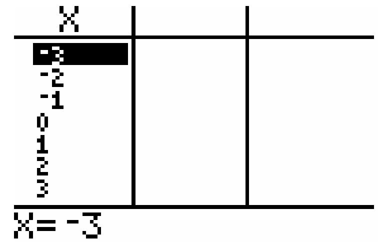
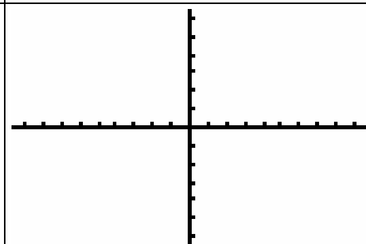
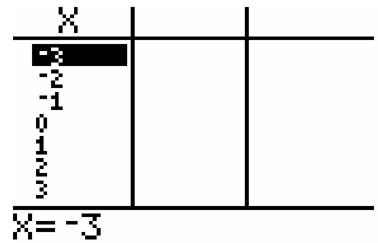
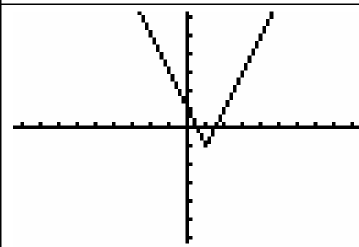
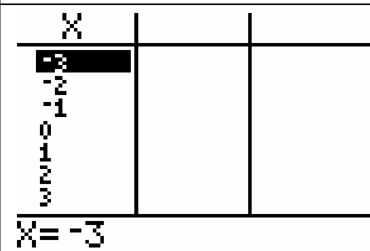
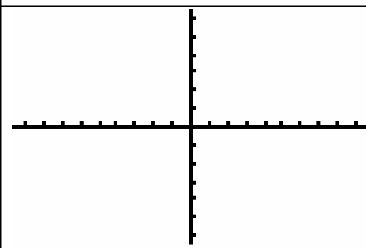
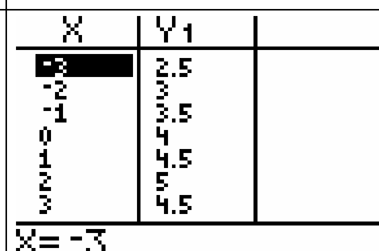
Write a generalization about how the parameter *a* affects the graph of the parent function.

Participant Pages: Transformations on the Parent Function: Changes to *h* and *k*

Function	**Graph**	**Table**	**Effect on Parent Function**
$y = \|x\| + 2$			
			
			The x-coordinate of the vertex is moved to the solution for $x + 2 = 0$. The graph of the parent function is shifted to the left by 2 units.
			

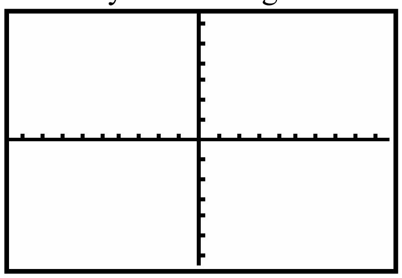
Write a generalization about how changes in the parameters *h* and *k* affect the graph of the parent function.

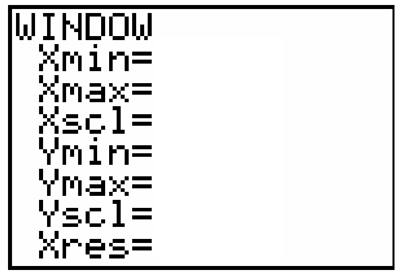
Participant Pages: Transformations to the Parent Functions: Changes to *a*, *h*, and *k*

Equation	Graph	Table	Effect on Parent Function
$y = \lvert x-1 \rvert + 2$			
			The vertex of the parent function has been translated to (–2, –4).
			
			
Write a generalization about how changes to the parameters *a, h, and k* affect the graph of the parent function.			

Participant Pages:
Solving Absolute Value Equations and Inequalities

1. Consider the system of equations $y = |x+3|$ and $y = 4$. Graph the system and sketch the graph. Describe your viewing window.



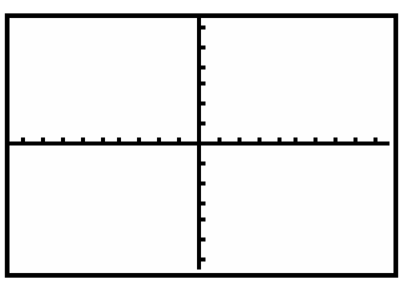


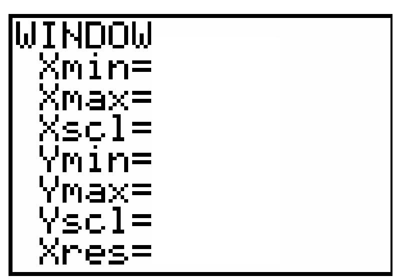
   a) What are the domain and range of each function in this system?

   b) What are the coordinates of the points that are solutions for this system? Why are there two solutions?

   c) How can you use the concept of substitution to write this system as one equation?

2. Graph the functions $y = x+3$, $y = -(x+3)$ and $y = 4$ and sketch the graph. Describe your viewing window.



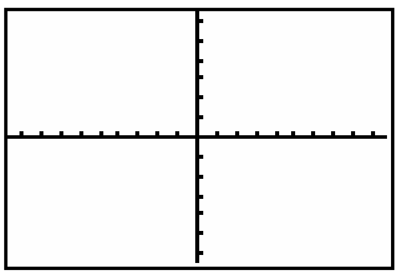


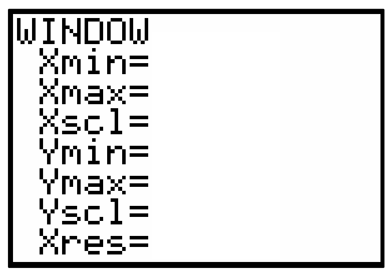
   a) What are the domain and range for each function in this system? How does this system of equations compare to the original system?

b) What are the coordinates of the points that are solutions for this system? How do these solutions compare to the solutions of the original system?

c) How can you use the concept of substitution to write this system of three equations as a system of two equations?

3. Graph the functions $y = x + 3$, $y = 4$, and $y = -4$. Graph this system and sketch the graph. Describe your viewing window.





a) What are the domain and range for each function in this system?

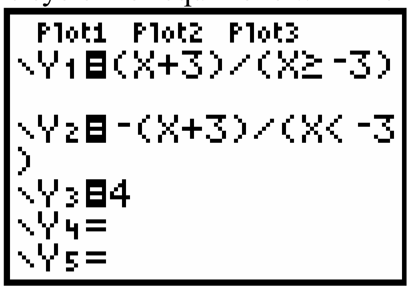
b) What are the coordinates of the points that are solutions for this system? How do the solutions for the graph above compare to the original solutions in question 1?

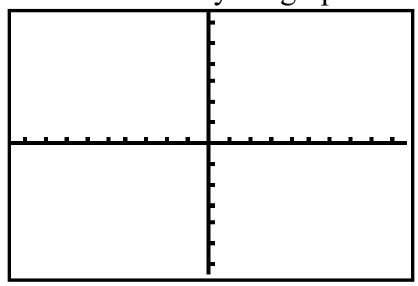
c) Compare the graph above and the graph in question 2 to the graph of the original system. Which graph is conceptually related to the graph of the original system? Which graph is not conceptually related? Why?

d) What misconceptions might arise by setting up a process to solve $x + 3 = 4$ and $x + 3 = -4$?

e) What restrictions do we need to place on the domains of the functions $y = x + 3$ and $y = -(x + 3)$ so that their graphs match the graph of the function $y = |x + 3|$?

f) Graph the system of equations with the restrictions. Sketch your graph.

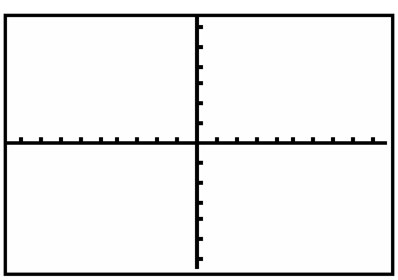


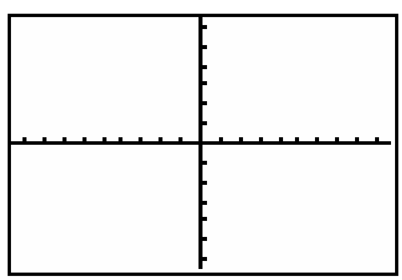


g) How does the graph of this system of equations and its solutions compare to the graph and solutions of $y = |x - 1|$ and $y = 2$ in question 4a?

4. Write a system of three equations that conceptually relate to the system of equations $|x - 1| = 2$.

a) Graph the system you wrote and compare it to the graph of equations $y = |x - 1|$ and $y = 2$. How do the graphs compare? How do the solutions compare?

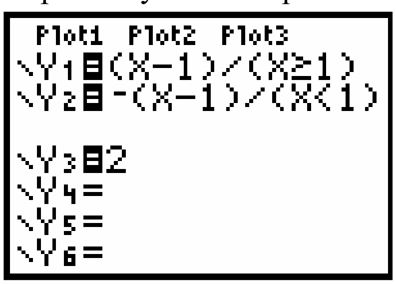


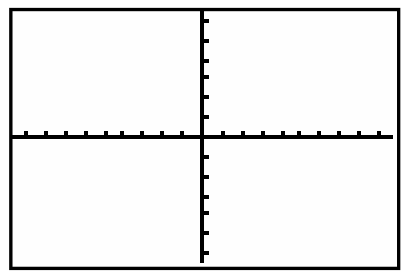


b) How do the equations $x-1=2$ and $-(x-1)=2$ relate to the graphs of the above systems?

c) What restrictions should we place on the domains of the functions in your system so that the graph of your system matches the graph of $y=|x-1|$?

d) Graph the system of equations with the restrictions. Sketch your graph.

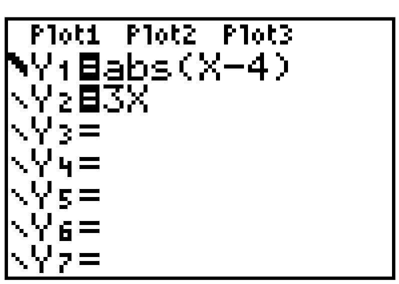


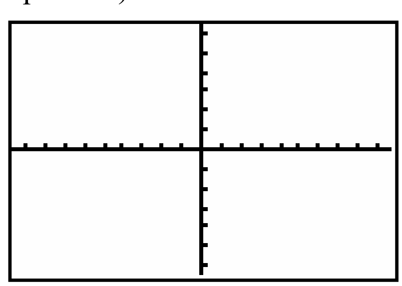


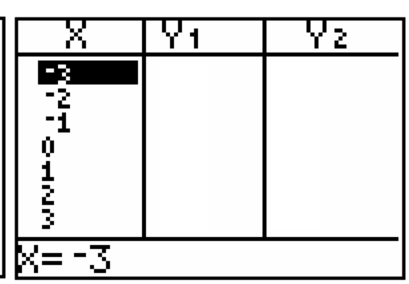
e) How does the graph of this system and its solutions compare to the graph of $y=|x-1|$ and $y=2$ in question 4a?

5. Write a statement about the connection between a system of equations such as $|x+2|=5$ and the system $x+2=5$ and $-(x+2)=5$.

6. Consider the system of equations represented by the equation $|x-4|=3x$. Graph the equations $y=|x-4|$ and $y=3x$, sketch the graph, and complete the table. (Hint: You may want to bold the absolute value equation.)

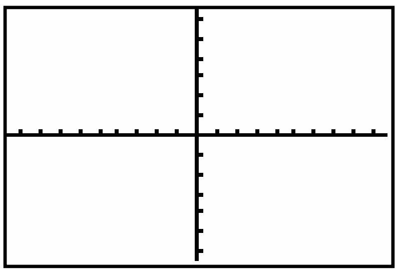
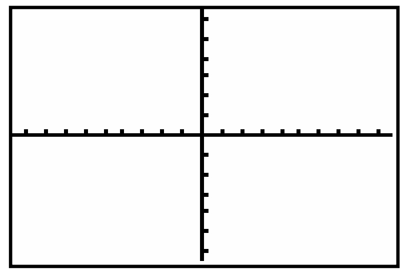






a) How many solutions does this system have? Why?

b) Compare the following systems in the table graphically, tabularly, and algebraically. (Hint: you may want to bold the equation(s) representing the absolute value function.)

<table>
<tr><th>System A</th><th>System B</th></tr>
<tr><td>$y = x - 4$
$y = -(x - 4)$
$y = 3x$</td><td>$y = x - 4$
$y = 3x$
$y = -3x$</td></tr>
<tr><td></td><td></td></tr>
<tr><td>
<table>
<tr><th>x</th><th>y_1</th><th>y_2</th><th>y_3</th></tr>
<tr><td>–3</td><td></td><td></td><td></td></tr>
<tr><td>–2</td><td></td><td></td><td></td></tr>
<tr><td>–1</td><td></td><td></td><td></td></tr>
<tr><td>0</td><td></td><td></td><td></td></tr>
<tr><td>1</td><td></td><td></td><td></td></tr>
<tr><td>2</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td></td><td></td><td></td></tr>
</table>
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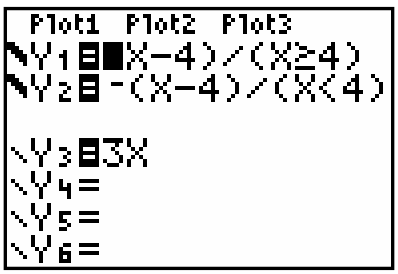
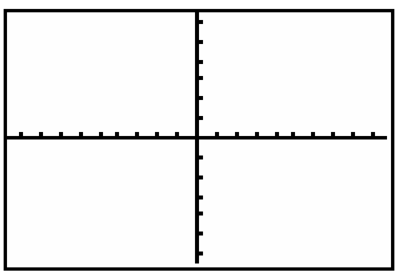
<table>
<tr><th>x</th><th>y_1</th><th>y_2</th><th>y_3</th></tr>
<tr><td>–3</td><td></td><td></td><td></td></tr>
<tr><td>–2</td><td></td><td></td><td></td></tr>
<tr><td>–1</td><td></td><td></td><td></td></tr>
<tr><td>0</td><td></td><td></td><td></td></tr>
<tr><td>1</td><td></td><td></td><td></td></tr>
<tr><td>2</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td></td><td></td><td></td></tr>
</table>
</td></tr>
<tr><td></td><td></td></tr>
</table>

c) What are the solutions for System A? What are the solutions for System B? How do those solutions compare to the original system?

d) Which system, A or B, conceptually relates to the original system?

e) Restrict the domains for the functions in System A, graph the new system and complete the table below. Find the table values for the solutions.





x	y_1	y_2	y_3
–3			
–2			
–1			
0			
1			
2			
3			

f) How are the solutions for System A without the restricted domain and System A with the restricted domain alike and different?

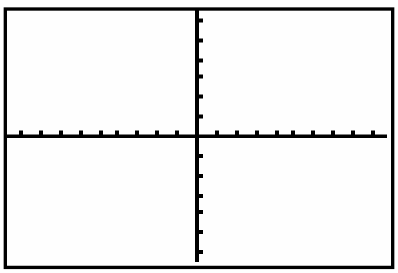
g) Why does System A without the restricted domain produce two solutions? Which solution of A is not a solution of the original system? What do we call that solution?

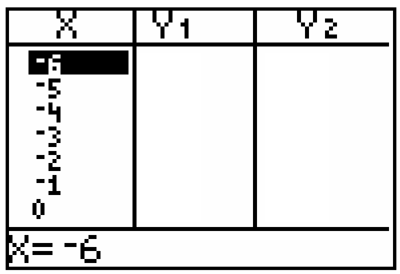
7. What understanding does graphing a system involving absolute value equations provide with regard to the actual number of solutions to the system and the corresponding equations that intersect? How does the graphical solution connect to the algebraic process?

8. Write a statement comparing the common algebraic process for solving absolute value equations to the conceptual understanding of the solutions to a system of absolute value equations.

9. Consider the system $|x+3|>4$.

   a) Show the solution graphically, tabularly, and symbolically.





   b) When we ask students to show us where $|x+3|>4$, what are we asking?

Leader Notes: How Much Will It Bend?

Purpose:

The purpose of this section of the professional development is to investigate stumbling blocks and challenges that students and teachers have in developing concepts and procedures involved with learning about rational functions.

Descriptor:

Explore phase 4 has four parts. The first part is an investigation of inverse variation by measuring deflection in various size bundles of linguine cantilevers. The second part looks at transformations to the reciprocal function $f(x)=\frac{1}{x}$. The third part investigates the rational function definition and connections between rational function form, factored form, transformation form, asymptotes of rational functions, and graphs of rational functions. The fourth part explores solving rational equations in a real work context.

Duration:

2.5 hours

TEKS:

a5	Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.
a6	Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.
2A.10	**Rational functions**. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.
2A.10A	The student is expected to use quotients of polynomials to describe the graphs of rational functions, predict the effects of parameter changes, describe limitations on the domains and ranges, and examine asymptotic behavior.
2A.10B	The student is expected to analyze various representations of rational functions with respect to problem situations.

2A.10C The student is expected to determine the reasonable domain and range values of rational functions, as well as interpret and determine the reasonableness of solutions to rational equations and inequalities.

2A.10D The student is expected to determine the solutions of rational equations using graphs, tables, and algebraic methods.

2A.10E The student is expected to determine solutions of rational inequalities using graphs and tables.

2A.10F The student is expected to analyze a situation modeled by a rational function, formulate an equation or inequality composed of a linear or quadratic functions, and solve the problem.

2A.10G The student is expected to use functions to model and make predictions in problem situations involving direct and inverse variation.

TAKS™ Objectives Supported:

While the Algebra II TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance: Copies of participant pages, cut out cards for the **Rational Function Card Sort**, cut out cards for the **Rational Function Card Match**

Presenter Materials: Overhead graphing calculator

Per group: String, ruler, masking tape, clear packing tape, linguine, 35mm film canister or candy container that is a cylinder, pennies (approximately 5), deflection grid (optional)

Per participant: Copy of participant pages, graphing calculator

Explore

Part 1: Linguine Cantilever

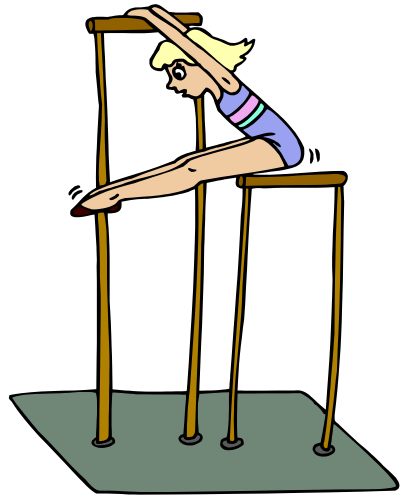
Leader Notes:
The purpose of Part 1 is to have participants actively involved in collecting data related to the reciprocal parent function. At the end of Part 1 the participants will be able to see how collecting data reinforces the understanding of the concept of the reciprocal parent function.

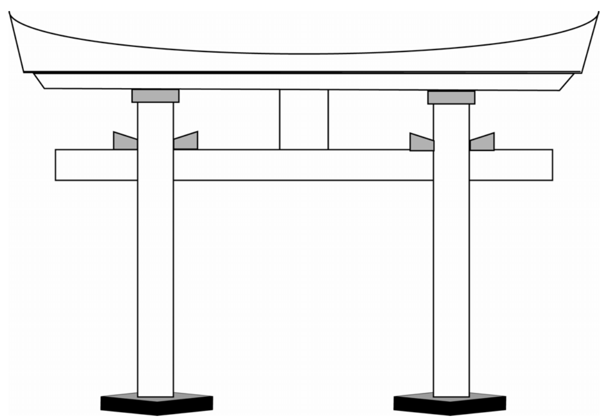
Ask participants to discuss how the objects on the first page are alike and how they are different. The idea you would like to elicit from the participants is that some of the objects are supported on two or more places of a beam and others are only supported at one end. Then proceed to discuss the porches on Fallingwater that are supported at only one end.

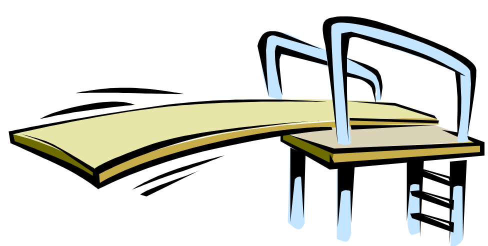
In this phase of the professional development participants investigate the reciprocal function generated by modeling the amount of deflection from a cantilevered beam. In this simplified experiment we are using linguine. It is not necessary that you use linguine, spaghetti or fettuccine will also work. If 35mm film canisters are not readily available, you can use pennies placed in a baggie, secured with transparent tape, and hung with an unfolded paper clip as your load. All participants should use the same type of linguine from a freshly opened package if you wish to average the data from the groups.

You may want to search for websites with information about cantilevers. For example, Wikipedia, the new observation deck for the Grand Canyon, or Fallingwater information.

How are the objects below alike? How are they different?





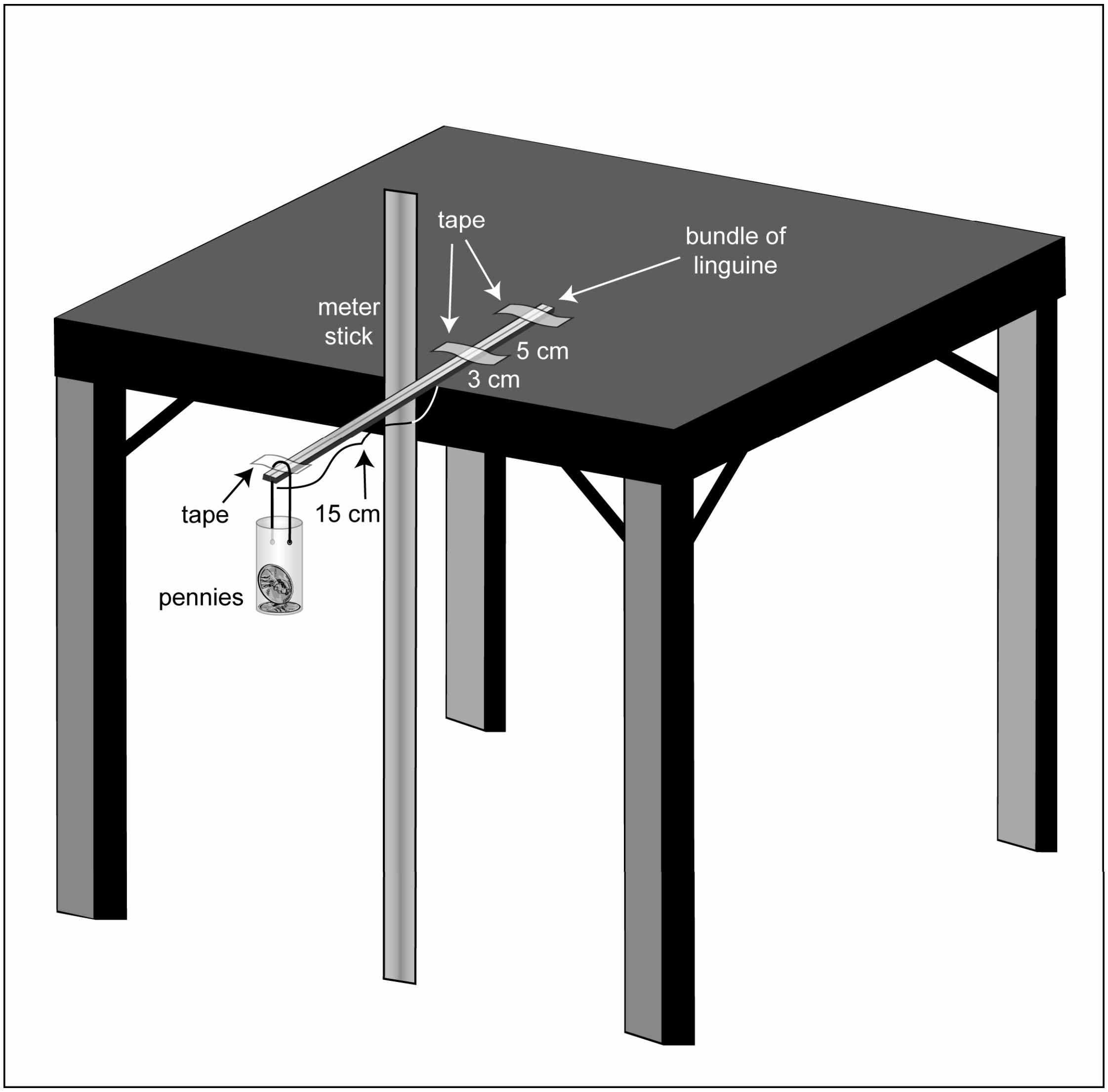


A cantilever is a projecting structure that is secured at only one end and carries a load on the other end. Diving boards and airplane wings are examples of horizontal cantilevers. Flagpoles and chimneys are vertical cantilevers. One of the most famous examples of a cantilever in architecture, which is shown below, is the Frank Lloyd Wright designed home, Fallingwater. The strength of a cantilever can be affected by variables such as length, load, cross sectional area, temperature, or elasticity. In this activity, you will be investigating the relationship between the thickness of a cantilever and the deflection in the cantilever when weight is added at the end.

In this investigation, you will keep the length of a piece of linguine that is hanging over the edge of a desk constant as you collect data on how much the linguine deflects. Deflection is the measure of the amount that the linguine bends in the downward direction. The number of pieces of linguine will change. Since you want to keep all variables (except for the ones you are investigating) constant, make sure to pay attention to the hints listed with the instructions.

Divide into groups of 3. Each person in the group has a job.

Materials manager:	Get the necessary materials, direct the team in setting up the investigation
Measures manager:	Measure the amount of deflection as the investigation proceeds
Data manager:	Record the necessary measurements in the table, share the data with the team



Data Collection Set-up Instructions

Step 1. **The materials person should get the necessary materials and begin to make bundles of linguine. Each bundle of 1, 2, 3, 4, 5, 6, 7, and 8 pieces of linguine should be taped one inch from each end. Since linguine is not all exactly the same length, try to keep one end of the bundle lined up.**

Step 2. **Tape a short piece of string to the 35mm film canister to form a handle. If you do not have a film canister, use a baggie, transparent tape, and a paper clip to build a weight to hang from the linguine.**

Step 3. **Tape one piece of linguine with 15 centimeters hanging over the edge of a desk. Put one piece of tape approximately 3 cm from the edge of the table. Place a second piece of tape over the end of linguine. Place the load (film canister) on the end of the linguine that is hanging over the edge of the desk. Slowly place pennies in the film canister until the linguine breaks. Wait 15 seconds before adding an additional penny. Use one less penny than the number required to break one piece of linguine as the load in your bucket for the remainder of this data collection experiment.**

Step 4. **Tape a meter stick perpendicularly to the floor next to a desk.**

Step 5. **Measure the linguine's height above the floor without the film canister attached. (Hint: It is easier to consistently measure the height using the bottom of the linguine.)**

Step 6. **Place your pennies into the bucket. (Hint: Place the pennies gently, throwing pennies into the bucket will alter the results.)**

Step 7. **Place the bucket on the end of the linguine that is hanging over the edge of the desk. (Hint: Place the string at the same point on the linguine for each trial. Use a piece of masking tape to hold the bucket onto the linguine.)**

Step 8. **Wait 15 seconds. Measure the amount of deflection in the linguine. (Hint: An easy method for measuring deflection is to use the eraser end of a pencil to line up the deflection of the end of the linguine with its measure on the meter stick.) Record your measurements in the table.**

Step 9. **Repeat the procedure with two pieces of linguine taped together still hanging 15 centimeters over the edge of the desk. Measure the deflection of the bundle of linguine.**

Step 10. **Continue repeating the procedure with additional pieces of linguine until you measure deflection with eight pieces taped together. Continue to record your data.**

1. **Fill in the table with the data you collected.**

Sample response:

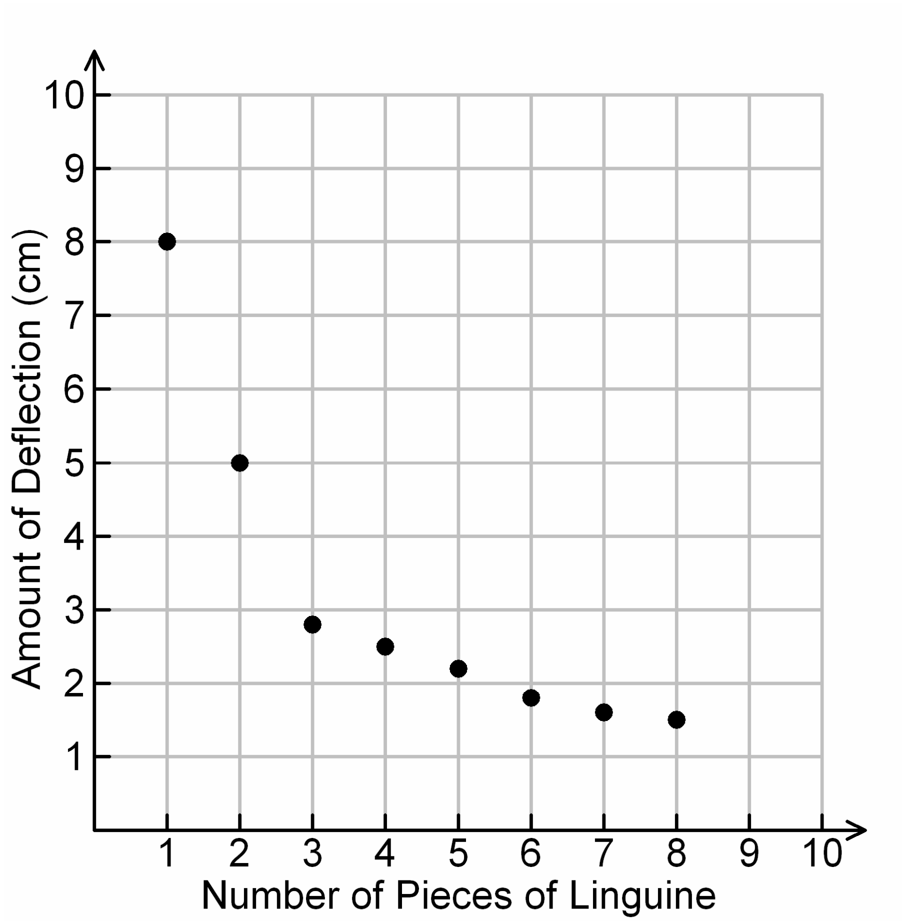
Number of Pieces of Linguine in the Bundle (x)	Starting Height of Linguine Bundle Above the Floor	Height of Linguine Bundle Above the Floor After the Load is Placed	Amount of Deflection in the Linguine (y)	Product of x and y ($x \cdot y$)
1	*74 cm*	*66 cm*	*8 cm*	*8*
2	*74 cm*	*69 cm*	*5 cm*	*10*
3	*74 cm*	*71.2 cm*	*2.8 cm*	*8.4*
4	*74 cm*	*71.5 cm*	*2.5 cm*	*10*
5	*74 cm*	*71.8 cm*	*2.2 cm*	*11*
6	*74 cm*	*72.2 cm*	*1.8 cm*	*10.8*
7	*74 cm*	*72.4 cm*	*1.6 cm*	*11.2*
8	*74 cm*	*72.5 cm*	*1.5 cm*	*12*

2. **Write a dependency statement relating the two variables.**
The deflection of the linguine bundles depends on the number of pieces of linguine in the bundle.

3. **What is a reasonable domain for the set of data?**
Responses may vary. The number of pieces of linguine might vary from 1 to 10.

4. **What is a reasonable range for the set of data?**
Responses may vary. The deflection appears to be between 1 and 9.

5. Make a scatterplot of the data you collected.



6. Verbally describe what happens in this data collection investigation.
As the number of pieces of linguine increases the deflection gets smaller.

7. Is this data set continuous or discrete? Why?
The data set is discontinuous; it is very difficult to use a fraction of a piece of linguine.

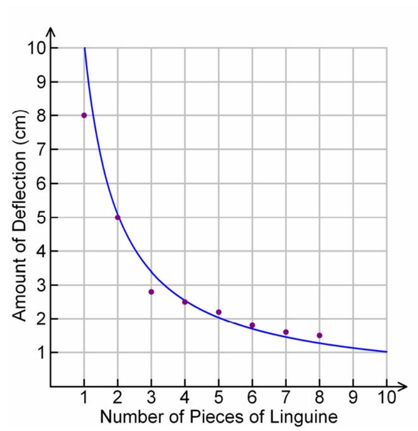
8. Does the set of data represent a function? Why?
Yes, the data set represents a function. For each bundle of linguine, there is one measured deflection value.

9. Does the data appear to be a linear, quadratic, exponential or some other type of parent function? Why do you think so?
The data does not appear to be linear or quadratic. It might appear to be exponential. If you investigate successive quotients, they are not congruent. So, this appears to be a new type of parent function.

10. Is the function increasing or decreasing?
The function values are decreasing, which makes sense since the more linguine in the bundle, the less deflection there appears to be.

11. Is the rate of change constant for this set of data?
The rate of change is not constant for this set of data. The rate of change varies depending on which pairs of points you investigate.

12. Determine a function rule that models the set of data you collected.
If you find the average of the products of x and y, you will have a close approximation to a in the function $y = \frac{a}{x}$. *Using the sample date, the function that models the data is approximately* $y = \frac{10.075}{x}$.



13. To get a better model add your set of data to the data of the entire group. Each group should send their data manager to the overhead to fill in the data collected for their group. Record the additional data in the table below. Find the average deflection for each bundle of linguine for the entire group.

Number of Pieces of Linguine in the Bundle	Amount of Deflection for Each Team												Average
	A	B	C	D	E	F	G	H	I	J	K	L	
1													
2													
3													
4													
5													
6													
7													
8													

14. Using the entire group's data, what function would you now use to model this situation?
Responses will vary. Hopefully, the data collected from the entire group will be a better approximation of the data.

15. How does this investigation connect to the TEKS from previous courses?
Responses may vary. Students should have investigated inverse variation in Algebra I. TEKS A.11B states, "The student is expected to analyze data and represent situations involving inverse variation using concrete models, tables, graphs, or algebraic methods."

As a whole group discuss the answer to question 16 before moving on to Part 2.

16. What are the key points students need to understand about the Linguine Cantilever before continuing the investigation of rational functions?
Responses may vary. Students need to understand that the product of the two variables is a constant in an inverse variation situation. They should be able to distinguish the inverse variation situation from previous parent functions they have studied. The average of the entire class's data may be a better model for the situation than the data from one group. Students are remembering what they have learned previously about inverse variation.

Part 2. Transformations to $f(x)=\frac{1}{x}$

Leader notes:
The purpose of Part 2 is to emphasize the fact that the reciprocal parent function is related to the linear parent function. At the end of this part, participants should recognize the need to be more overt in demonstrating how the function values in the table change, in addition to changes to the graph during vertical dilations, vertical shifts, and horizontal shifts to the parent function. Explain to participants that they will be investigating a parent function new to Algebra 2 students.

1. **What is the reciprocal of the linear parent function,** $f(x)=x$**?**

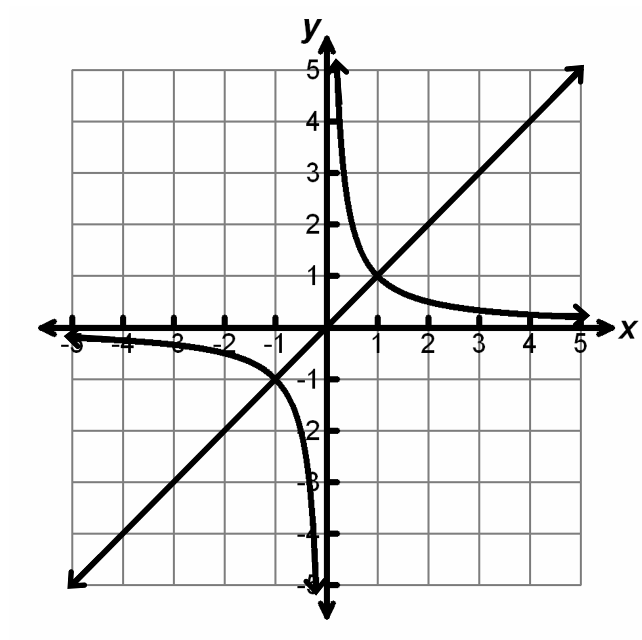
   The reciprocal is $g(x)=\frac{1}{x}$.

2. **Let's investigate some of the attributes of the function and its reciprocal. Fill in the tables with several values for each function. Draw a sketch of the graphs of the two functions on the same set of axes.**

   Sample response:

$f(x)=x$	
x	***y***
–3	*–3*
–2	*–2*
–1	*–1*
–0.5	*–0.5*
–0.1	*–0.1*
0	*0*
0.1	*0.1*
0.5	*0.5*
1	*1*
2	*2*
3	*3*

$f(x)=\frac{1}{x}$	
x	***y***
–3	*–1/3*
–2	*–1/2*
–1	*–1*
–0.5	*–2*
–0.1	*–10*
0	*0*
0.1	*10*
0.5	*2*
1	*1*
2	*1/2*
3	*1/3*



3. **Using your graphing calculator (if necessary), fill in the tables below. Let $f(x) = x$ be Y_1, and let $g(x) = \frac{1}{x}$ be Y_2.**

$$Y_1 = x$$

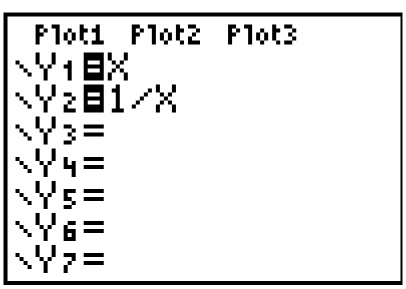
$$Y_2 = \frac{1}{x}$$

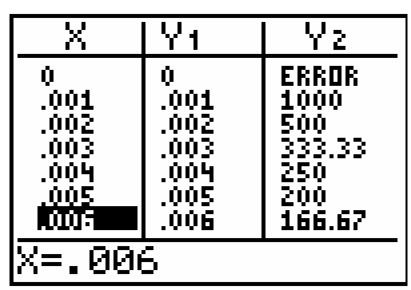
$Y_1 = x$		$Y_2 = \frac{1}{x}$
$(-\infty, \infty)$	**Intervals where the function is increasing**	*none*
none	**Intervals where the function is decreasing**	$(-\infty, 0) \cup (0, \infty)$
none	**Intervals where the function is undefined**	$x = 0$
$(0, 0)$	**Coordinates of the *x*-intercepts (zeros)**	*none*
none	**Equations of any asymptotes**	$x = 0$ $y = 0$

4. **What do you notice about the graphs of the linear parent function and its reciprocal?**
Whenever the linear function is increasing, the reciprocal is decreasing. The x-intercept of the linear function is one of the asymptotes of the reciprocal function.

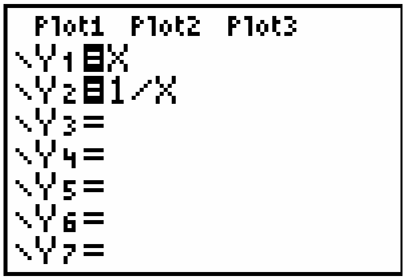
5. **Where do the linear parent function and its reciprocal intersect?**
They intersect at (1, 1) and (–1, –1).

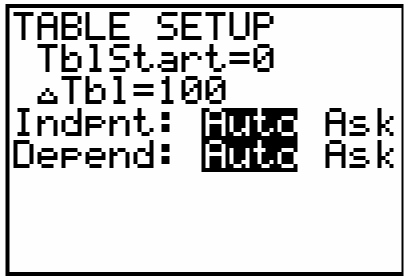
6. **How could you have your students investigate what happens to *f(x)* as *x* gets closer and closer to 0 using the graphing calculator?**
Responses may vary. Set the ΔTbl to a small number such as 0.001 and investigate the y values.

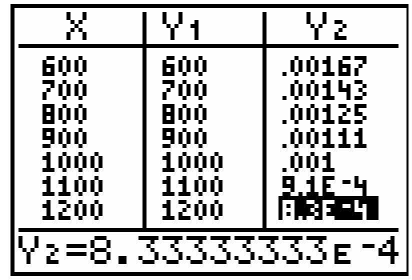




7. How could you have your students investigate what happens to *f(x)* as *x* gets larger and larger?
Actually look at the y values in the table as x gets larger. The y values get closer and closer to 0.







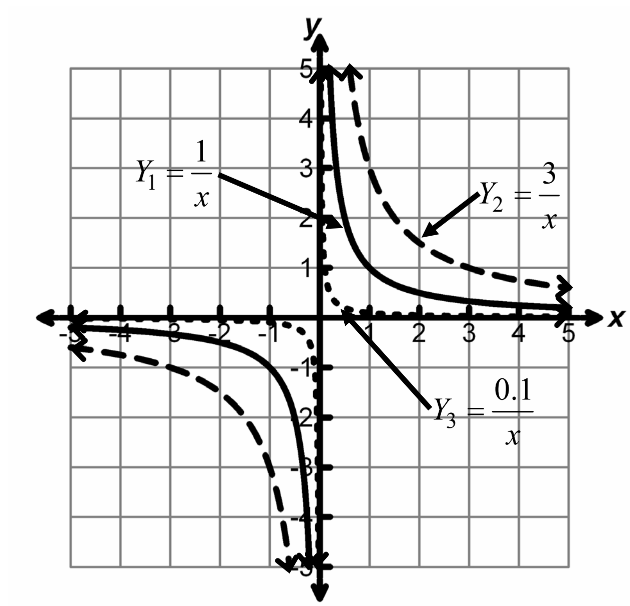
8. How do the Algebra II TEKS name this new parent function?
In the Algebra II TEKS this new parent function is the reciprocal function.

9. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x)=\frac{1}{x}$, when it is multiplied by a constant as in the examples below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results.

$Y_1=\frac{1}{x}$

$Y_2=\frac{3}{x}$

$Y_3=\frac{0.1}{x}$



Sample response:

x	$Y_1=\frac{1}{x}$	$Y_2=\frac{3}{x}$	$Y_3=\frac{0.1}{x}$
−3	−0.33…	−1	−0.03…
−2	−0.5	−1.5	−0.05
−1	−1	−3	−0.1
0	error	error	error
1	1	3	0.1
3	0.33…	1	0.033…
2	0.5	1.5	0.05
4	0.25	0.75	0.25

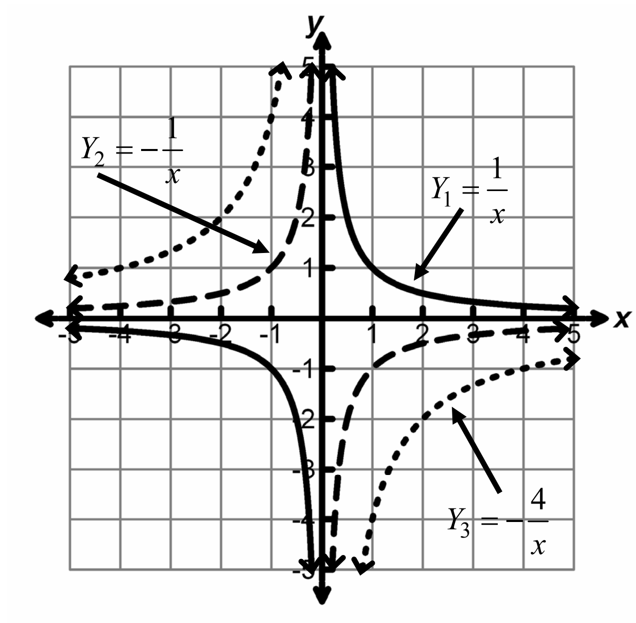
If the constant is larger than 1, the graph is stretched vertically. If the constant is less than one but greater than zero, the graph is vertically compressed.

10. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x) = \frac{1}{x}$, when it is multiplied by a negative constant as in the examples below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results which includes any asymptotes.

$Y_1 = \frac{1}{x}$

$Y_2 = -\frac{1}{x}$

$Y_3 = -\frac{4}{x}$



x	$Y_1 = \frac{1}{x}$	$Y_2 = -\frac{1}{x}$	$Y_3 = -\frac{4}{x}$
−3	−0.33…	0.33…	1.33…
−2	−0.5	0.5	2
−1	−1	1	4
0	error	error	error
1	1	−1	−4
2	0.5	−0.5	−2
3	0.33…	−0.33…	−1.33…
4	0.25	−0.25	−1

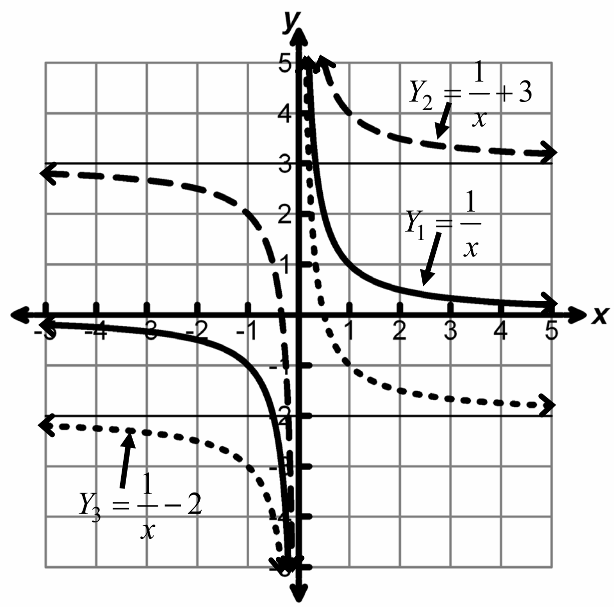
Sample response: The y-values in Y_2 and Y_3 are the opposite sign of the y-value in Y_1. So, it would appear that the graph of the parent function is reflected across the x-axis. All three functions still have "error" at x = 0, a vertical asymptote. The vertical stretching properties are the same. There is a horizontal asymptote at y = 0.

11. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x) = \frac{1}{x}$, if a constant is added to the function as in the three functions listed below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results which includes any asymptotes.

$Y_1 = \frac{1}{x}$

$Y_2 = \frac{1}{x} + 3$

$Y_3 = \frac{1}{x} - 2$



x	$Y_1 = \frac{1}{x}$	$Y_2 = \frac{1}{x} + 3$	$Y_3 = \frac{1}{x} - 2$
–3	*–0.3…*	*2.6…*	*–2.3…*
–2	*–0.5*	*2.5*	*–2.5*
−1	*−1*	*2*	*−3*
0	*error*	*error*	*error*
1	*1*	*4*	*–1*
2	*0.5*	*3.5*	*−1.5*
3	*0.3…*	*3.3…*	*–1.6…*
4	*0.25*	*3.25*	*−1.75*

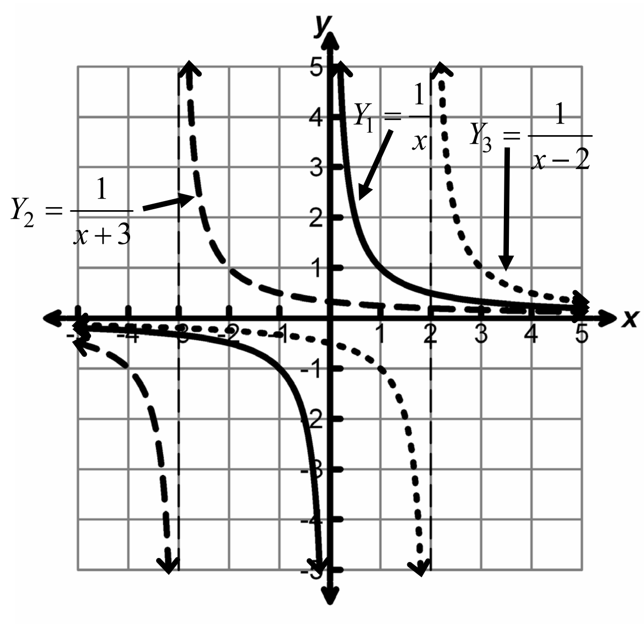
Sample response: Each y-value in Y_2 is greater than the corresponding Y_1 value by 3. Each y-value in Y_3 is 2 less than the corresponding y-value in Y_1. The graphs appear to slide up or down vertically. The horizontal asymptote is changing. The horizontal asymptote appears to occur at the value of the vertical shift. A vertical asymptote still occurs at x = 0.

12. Using your graphing calculator describe what happens to the reciprocal parent function, $g(x)=\frac{1}{x}$, if a constant is added to the *x*-coordinate in the denominator as in the three functions listed below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results which includes any asymptotes.

$Y_1=\frac{1}{x}$

$Y_2=\frac{1}{x+3}$

$Y_3=\frac{1}{x-2}$



x	$Y_1=\frac{1}{x}$	$Y_2=\frac{1}{x+3}$	$Y_3=\frac{1}{x-2}$
−3	−0.333…	error	−0.2
−2	−0.5	1	-0.25
−1	−1	0.5	−0.333…
0	error	0.333…	−0.5
1	1	0.25	−1
2	0.5	0.2	error
3	0.333…	0.166…	1
4	0.25	0.143	0.5

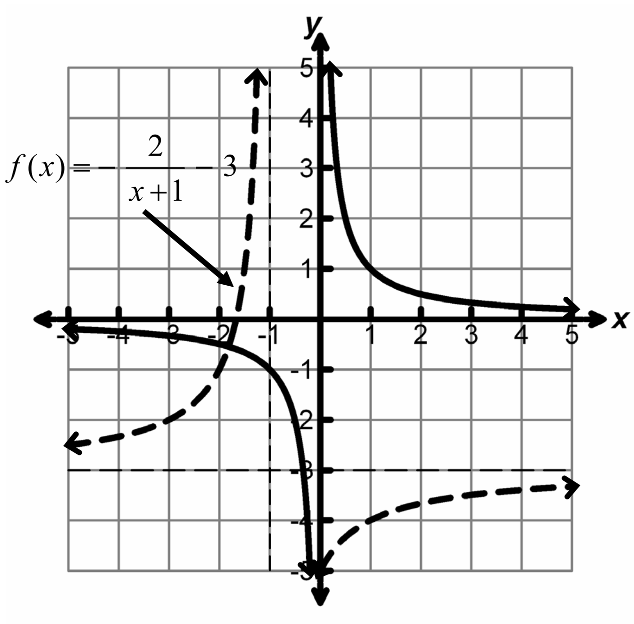
Possible response: The graphs appear to have been shifted horizontally left and right depending on the sign of the constant. Looking at the table, the y-values shift 3 units to the left or 2 units to the right, depending on the function. The horizontal asymptote for each of the functions is y = 0. The vertical asymptotes are changing depending on the value of the constant added to x.

13. Predict, describe and then sketch the transformations to the reciprocal parent function in the function below.

$$f(x)=-\frac{2}{x+1}-3$$

Possible response: The function will be reflected over the x-axis, stretched vertically by a factor of 2, slid to the left one unit, and slid down three units. The horizontal asymptote is y = −3, and the vertical asymptote is x = −1.

x	$Y_1 = \frac{1}{x}$	$Y_1 = -\frac{2}{x}$	$Y_1 = -\frac{2}{x+1}$	$Y_2 = -\frac{2}{x+1} - 3$
−3	*−0.333…*	*0.66…*	*1*	*−2*
−2	*−0.5*	*1*	*2*	*−1*
−1	*−1*	*2*	*error*	*error*
0	*error*	*error*	*−2*	*−5*
1	*1*	*−2*	*−1*	*−4*
2	*0.5*	*−1*	*−0.66…*	*−3.66…*
3	*0.333…*	*−0.66…*	*−0.5*	*−3.5*
4	*0.25*	*−0.5*	*−0.4*	*−3.33…*



14. Using the variables *a*, *h*, and *k* describe how transformations to the reciprocal function are similar to transformations to other parent functions.
Transformations to parent functions have the same properties. This helps students connect their new knowledge to previous concepts.

As a whole group discuss the answer to question 15 before moving on to Part 3.

15. What are the key points students need to understand about transformations to the reciprocal function before continuing the investigation of rational functions?
Responses may vary. Students need to understand that the reciprocal parent function is affected by transformations in the same way as the quadratic parent function. The transformations can be justified using a graph, an equation, or a table.

<u>Part 3</u>: Rational Functions

Leader notes:
The purpose of Part 3 is to have participants clarify their understanding of the definition of a rational function.

1. **Give each pair of participants a set of the cards, *Rational Function Card Sort*. Have them sort the functions into two (and only two) groups. Try not to give any hints ahead of time that some of the functions are rational functions and some are not. After all of the groups have sorted their cards, encourage them to describe their sorting method.**

   The rational functions are b(x), d(x), h(x), j(x), k(x), n(x), p(x), q(x), t(x), w(x), y(x), z(x). Functions that are not rational are c(x), f(x), g(x), m(x), r(x), s(x), u(x), and v(x).

Facilitation Questions

- **What is the definition of a rational function?**
  A rational function is any expression that can be written in the form $\frac{p(x)}{q(x)}$ *where p(x) and q(x) are both polynomials and q(x) ≠ 0.*

- **What is the definition of a polynomial?**
  A polynomial in one variable is any expression that can be written in the form
  $$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$$
  where x is a variable, the exponents are nonnegative integers, and the coefficients are real numbers. (Discovering Advanced Algebra, p. 360)
  Some definitions allow coefficients to be complex numbers.
  "Polynomials have no variables in denominators or exponents, no roots or absolute values of variables, and all variables have whole number exponents." (Holt, Algebra 2)

- **How does what students have learned previously about rational numbers relate to their new knowledge of rational functions?**
  Students learned about fractions (rational numbers) in elementary and middle school. They learned about inverse variation in Algebra 1. Now, the reciprocal parent function is introduced as a subset of the larger set of rational functions.

- **What are the most interesting parts of rational functions?**
  Responses may vary. A possible response might be: "The most interesting parts are the vertical and horizontal asymptotes and holes."

- **Which polynomials are Algebra II students required to use in rational functions based on the Algebra II TEKS?**
  The TEKS require only that students use polynomials of degree 1 or 2 in the numerator and denominator of a rational function.

2. **After participants have completed the Rational Function Card Sort as a whole group discuss any differences in their sorting methods before proceeding.**

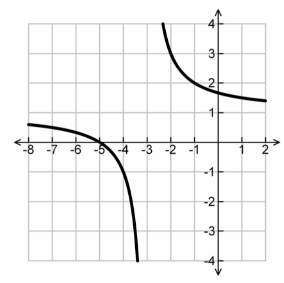
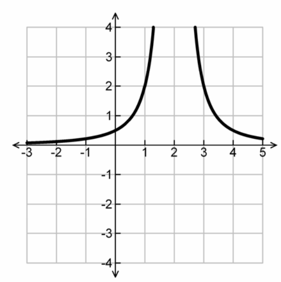
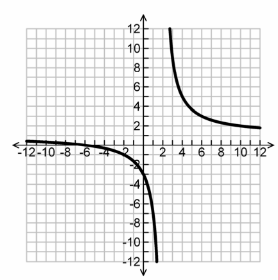
3. **Give each pair of participants a set of the cards, Rational Function Card Match. Have participants match the rational form of each rational function to the transformation form or factored form, the equations of any asymptotes, the coordinates of any removable discontinuities (holes), and the graph of the function.**

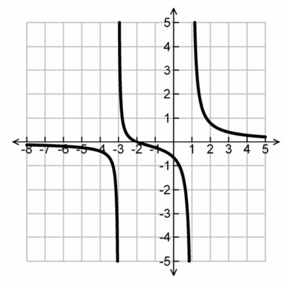
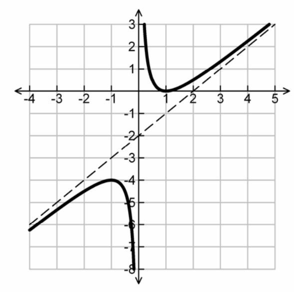
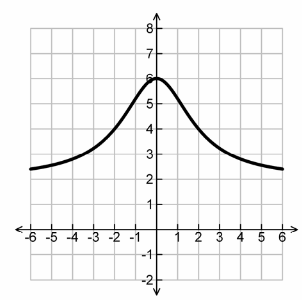
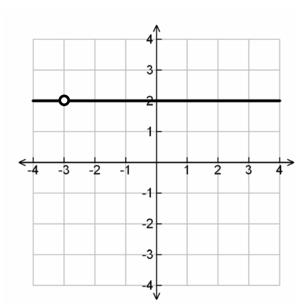
After participants have completed the Rational Function Card Match as a whole group discuss the answer to question 4 before moving on to Part 4.

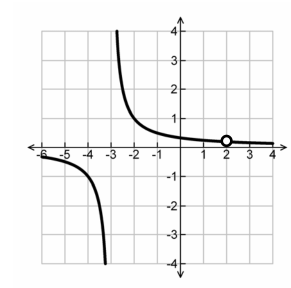
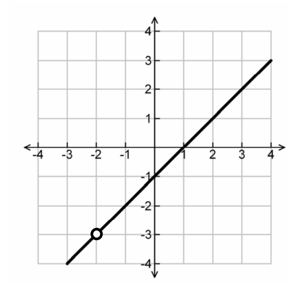
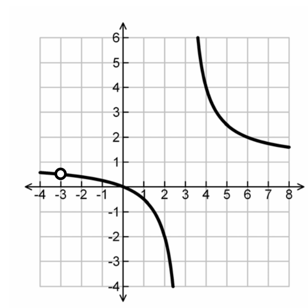
4. **What are the key points students need to understand about rational functions to be able to do the Rational Function Card Match?**
   Responses may vary. Students need to understand that rational functions can be written in several forms. One form might be good for finding the vertical asymptotes or x-intercepts. Another form might be good for finding an oblique asymptote. Students should be able to match the graph to its equations and identify any asymptotes or holes.

Rational Function Card Match

Answer Key

Rational Function Form	Transformation Form	Discontinuities, Asymptotes, Other Noteworthy Points	Graph
$f(x)=\dfrac{x+5}{x+3}$	$f(x)=\dfrac{2}{x+3}+1$	$x=-3$	
		$y=1$	
		x-intercept at (−5, 0)	
A		y-intercept at $(0, \frac{5}{3})$	
$f(x)=\dfrac{2}{x^2-4x+4}$	$f(x)=\dfrac{2}{(x-2)^2}$	$x=2$	
		$y=0$	
		no x-intercepts	
B		y-intercept at (0, 0.5)	
$f(x)=\dfrac{x+6}{x-2}$	$f(x)=\dfrac{8}{x-2}+1$	$x=2$	
		$y=1$	
		x-intercept at (−6, 0)	
C		y-intercept at (0, −3)	

$f(x)=\dfrac{x+2}{x^2+2x-3}$ **D**	$f(x)=\dfrac{x+2}{(x+3)(x-1)}$	$x=1$ and $x=-3$ $y=0$ x-intercept at $(-2, 0)$ y-intercept at $(0, -\frac{2}{3})$	
$f(x)=\dfrac{x^2-2x+1}{x}$ **E**	$f(x)=x-2+\dfrac{1}{x}$	$y=x-2$ $x=0$ x-intercept at $(1, 0)$ no y-intercepts	
$f(x)=\dfrac{2x^2+24}{x^2+4}$ **F**	$f(x)=\dfrac{16}{x^2+4}+2$	$y=2$ no vertical asymptotes no x-intercepts y-intercept at $(0, 6)$	
$f(x)=\dfrac{2x+6}{x+3}$ **G**	$f(x)=2,$ $x\neq -3$	no horizontal asymptote removable discontinuity (hole) at $(-3, 2)$ no x-intercepts y-intercept at $(0, 2)$	

$f(x)=\dfrac{x-2}{x^2+x-6}$ **H**	$f(x)=\dfrac{1}{x+3}$, $x \neq -2$	$x = -3$ and $y = 0$ removable discontinuity (hole) at (2, 0.2) no *x*-intercepts *y*-intercept at (0, $\frac{1}{3}$)	
$f(x)=\dfrac{x^2+x-2}{x+2}$ **I**	$f(x)=x-1$, $x \neq -2$	no horizontal asymptote removable discontinuity (hole) at (−2, −3) *x*-intercept at (1, 0) *y*-intercept at (0, −1)	
$f(x)=\dfrac{x^2+3x}{x^2-9}$ **J**	$f(x)=\dfrac{3}{x-3}+1$, $x \neq -3$	$x = 3$ and $y = 1$ removable discontinuity (hole) at (−3, 0.5) *x*-intercept at (0, 0) *y*-intercept at (0, 0)	

<u>Part 4</u>: Length of a Yellow Light

Leader notes:
The purpose of Part 4 is to allow participants to investigate a rational function in a real world context. At the end of the investigation they should be more willing to consider solving rational equations with tables and graphs in addition to the traditional symbolic solution.

One of the formulas traffic engineers use to help them calculate the length of time a traffic light should remain yellow is

$$Y(t) = t + \frac{v}{2a} + \frac{w+L}{v}.$$

The formula takes into account reaction time, braking time, and intersection clearance time. The variables used in this calculation are:

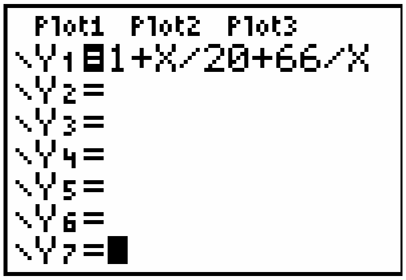
- t = reaction time (usually 1 second)
- v = velocity of the vehicle (in feet/second)
- a = deceleration rate (approximately 10 feet/ sec^2)
- w = width of the intersection (feet)
- L = length of the vehicle (feet)

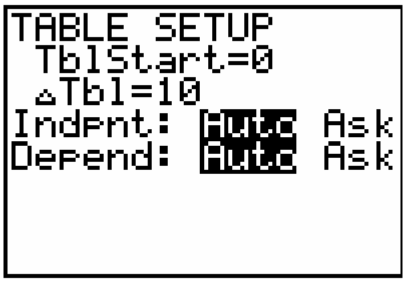
In order to calculate the speed limit for a certain intersection that is 48 feet wide, the engineer uses an average car length of 18 feet. She can calculate the length of time the traffic signal should remain yellow at that intersection based on the velocity in feet/second using the following formula:

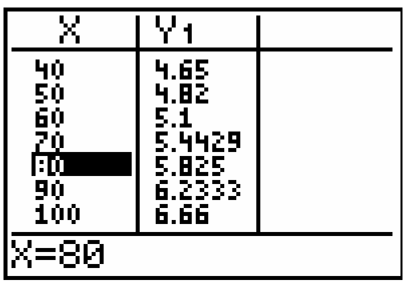
$$Y(v) = 1 + \frac{v}{20} + \frac{66}{v}.$$

1. **If the posted speed limit at the intersection is 55 miles per hour, how long should the signal remain yellow?**
   Change the 55 mph speed limit into feet per second:

$$\frac{55 \text{ miles}}{1 \text{ hour}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = \frac{80.667 \text{ feet}}{1 \text{ second}}.$$

<table>
<tr><td>Solution using a table:</td><td>Solution using a graph:</td><td>Symbolic solution:</td></tr>
<tr><td>






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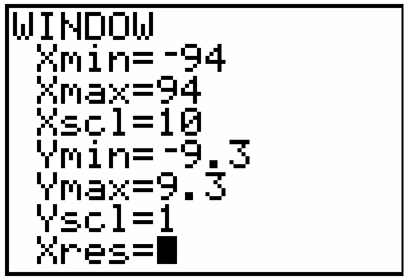

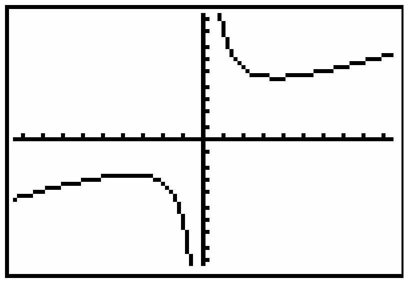
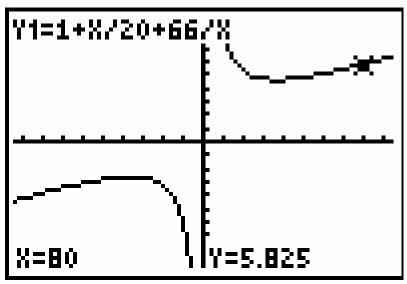

</td><td>
$Y(v)=1+\frac{80.7}{20}+\frac{66}{80.7}$,
find $Y(v)$ when $v=80.7$ ft/sec
so, $Y(80.7)\approx 5.85$ sec
</td></tr>
</table>

So, if the posted speed limit is 55 mph, then according to the formula, the traffic signal should remain yellow for approximately 6 seconds.

2. **For a traffic signal to remain yellow for 4 seconds, what should the department of transportation post as the speed limit?**
   According to the formula applied to this particular intersection, the traffic signal should never remain yellow for only 4 seconds.

3. **If the speed of vehicles at a particular intersection varies between 30 and 50 mph, how long do you think the traffic signal should remain yellow?**
   First change 30 mph to 20.45 ft/sec and 50 mph to 34.09 ft/sec. Looking at the table and the graph at 30 mph the light should remain yellow for 5.2 seconds and at 50 mph the light should remain yellow for 4.6 seconds. The best time for the signal to remain yellow might be 5.5 seconds.

4. A tractor trailer that is approximately 36 feet long travels through the same intersection when the signal remains yellow for 6 seconds. Based on the formula, how fast should the tractor trailer be allowed to drive through the intersection? How fast should a car be allowed to drive through the intersection?

$$Y(v) = 1 + \frac{v}{20} + \frac{84}{v}$$

The speed limit for tractor trailers at one time should be 78.64 ft/sec, or approximately 53 mph; the second time the signal remains yellow for 6 seconds is 21.36 ft/sec, or approximately 15 mph. The speed limit for cars at one time should be approximately 84.35 ft/sec, or approximately 57.5 mph; the second time the signal remains yellow for 6 seconds is 15.65 ft/sec, or 10.7 mph.

For tractor trailers,

```
 Plot1  Plot2  Plot3
\Y1=1+X/20+84/X
\Y2=6
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
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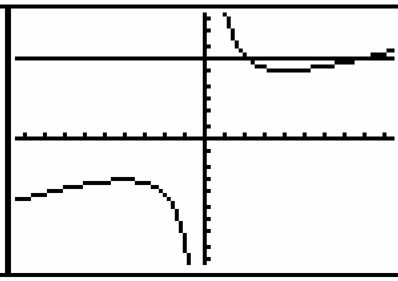
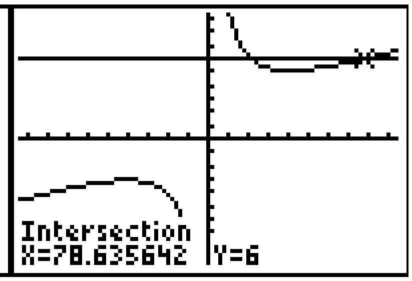
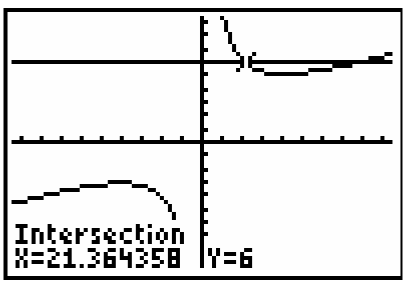






Using the table feature,

X	Y1	
74	5.8351	
75	5.87	
76	5.9053	
77	5.9409	
78	5.9769	
79	6.0133	
80	6.05	

X=78

X	Y1	Y2
17	6.7912	6
18	6.5667	6
19	6.3711	6
20	6.2	6
21	6.05	6
22	5.9182	6
23	5.8022	6

X=20

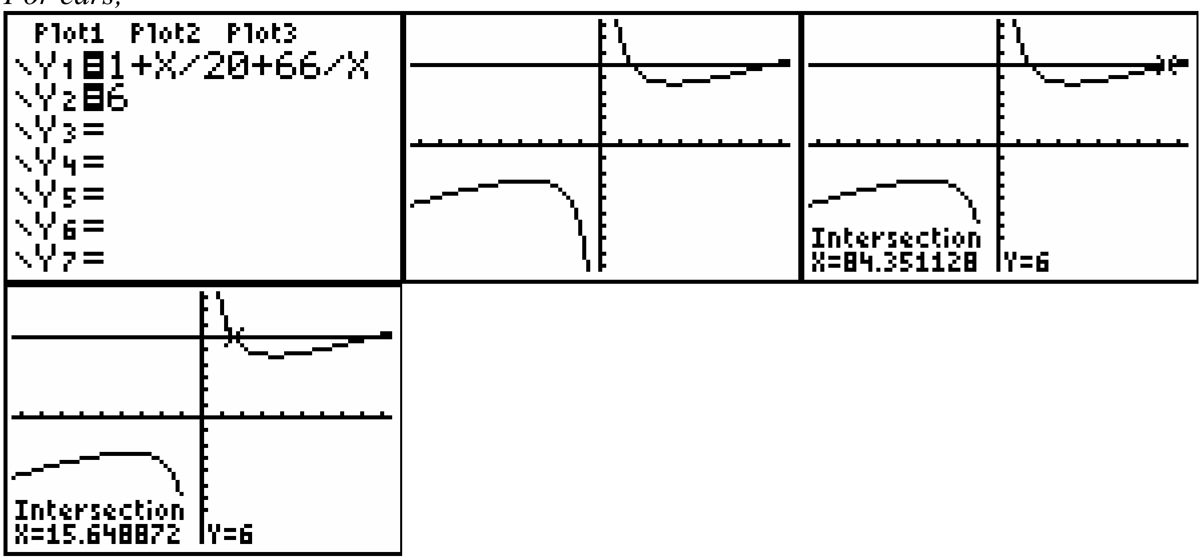
Symbolically,

$$
\begin{aligned}
6 &= 1+\frac{v}{20}+\frac{84}{v} \\
20v(6) &= \left(1+\frac{v}{20}+\frac{84}{v}\right)20v \\
120v &= 20v+v^2+1680 \\
0 &= v^2-100v+1680 \\
v &= \frac{100 \pm \sqrt{(-100)^2-4(1680)}}{2}
\end{aligned}
$$

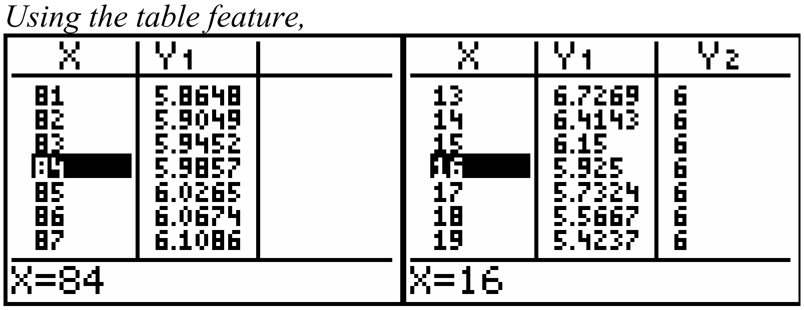
$v \approx 78.64$ ft/sec or $v \approx 21.36$ ft/sec

$v \approx 53$ mph or $v \approx 15$ mph

For cars,



Using the table feature,



X	Y1	
81	5.8648	
82	5.9049	
83	5.9452	
84	5.9857	
85	6.0265	
86	6.0674	
87	6.1086	

X=84

X	Y1	Y2
13	6.7269	6
14	6.4143	6
15	6.15	6
16	5.925	6
17	5.7324	6
18	5.5667	6
19	5.4237	6

X=16

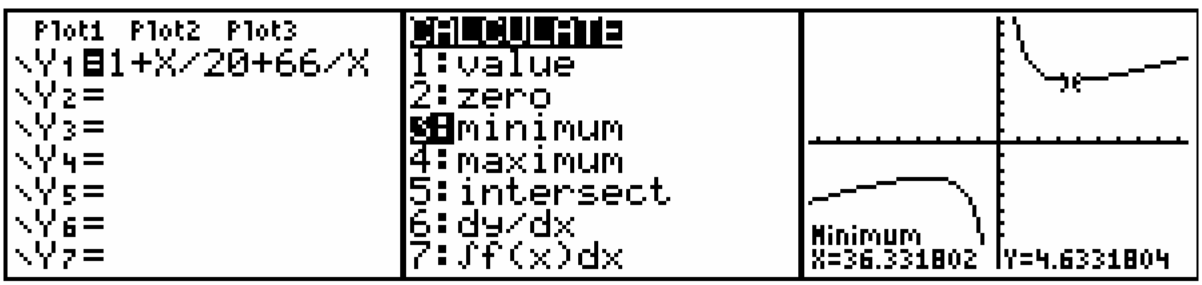
Symbolically,

$$6 = 1 + \frac{v}{20} + \frac{66}{v}$$
$$20v(6) = \left(1 + \frac{v}{20} + \frac{66}{v}\right)20v$$
$$120v = 20v + v^2 + 1320$$
$$0 = v^2 - 100v + 1320$$
$$v = \frac{100 \pm \sqrt{(-100)^2 - 4(1320)}}{2}$$
$$v \approx 84.35 \text{ ft/sec or } v \approx 15.65 \text{ ft/sec}$$
$$v \approx 57.5 \text{ mph or } v \approx 10.7 \text{ mph}$$

It is probably safer for the tractor trailers to travel approximately 5 miles per hour slower than the cars.

5. Do you think this is a good model for length of time that traffic signals should remain yellow for every *x* value in the domain?

Responses may vary. Some participants may think that the model only seems to work well for speeds larger than 36 ft/sec, or approximately 25 mph, at this particular intersection. The rational function approaches an asymptote at x = 0, so for extremely slow speeds the length of the yellow light is very long.



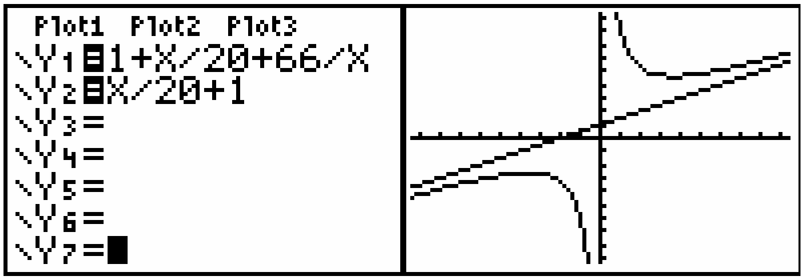
Bonus question:

6. What is the oblique asymptote for this rational function? Graph both the function and the asymptote on your graphing calculator.

$$Y(v) = 1 + \frac{v}{20} + \frac{66}{v}$$

As v gets larger and larger, $\frac{66}{v}$ *approaches 0. So, the equation of the oblique asymptote is*

$Y(v) = 1 + \frac{v}{20}$.



After participants have completed the Length of a Yellow Light as a whole group discuss the answer to question 7 before moving on to the Explain.

7. **Do you think that students should solve every problem involving rational functions symbolically? What understanding would students gain from solving with tables and graphs?**
   Responses may vary. Students need to have as many different possible ways to solve a problem as they can. Not only can they check their answer by solving differently, but they may be able to conceptually understand their solution when they can see it in a graph or in a table.

Explain

Leaders' Note: The Maximizing Algebra II Performance (MAP) professional development is intended to be an extension of the ideas introduced in Mathematics TEKS Connections (MTC). Throughout the professional development experience, we will allude to components of MTC such as the Processing Framework Model, the emphasis of making connections among representations, and the links between conceptual understanding and procedural fluency.

Debriefing the Experience:

1. **What concepts did we explore in the previous set of activities? How were they connected?**
   Responses may vary and may include generating rational functions to describe data and transforming rational functions.

2. **What procedures did we use to work with rational functions and equations? How are they related?**
   Tabular, graphical, and symbolic procedures (including procedures to solve rational equations) were all used throughout the Explore phase. Ultimately, they are all connected through the numerical relationships used to generate them.

3. **What knowledge from Algebra I do students bring about rational functions?**
   According to the Algebra I TEKS, students model inverse variation using rational functions.

4. **After working with rational functions in Algebra II, what are students' next steps in Precalculus or other higher mathematics courses?**
   According to the Precalculus TEKS, students will continue their study of continuity and end-behavior, asymptotes, and limits.

Anchoring the Experience:

5. **Distribute to each table group a poster-size copy of the Processing Framework Model.**

6. **Ask each group to respond to the question:**

   Where in the processing framework would you locate the different activities from the Explore phase?

7. **Participants can use one color of sticky notes to record their responses.**

Horizontal Connections within the TEKS

8. **Direct the participants' attention to the second layer in the Processing Framework Model: Horizontal Connections among Strands.**

9. **Prompt the participants to study the Algebra II TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.**

10. Invite each table group to share 2 connections that they found and record them so that they are visible to the entire group.

Vertical Connections within the TEKS

11. Direct the participants' attention to the third layer in the Processing Framework Model: Vertical Connections across Grade Levels.

12. Prompt the participants to study the Algebra I, Geometry, Math Models, and Precalculus TEKS and record those TEKS that connect to this Explore/Explain cycle. Prompt participants to attend to both the knowledge statements as well as the student expectations.

13. Invite each table group to share 2 connections that they found, recording so that the entire large group may see.

14. Provide each group of participants with a clean sheet of chart paper. Ask them to create a "mind map" for the mathematical term "rational functions."

15. Provide an opportunity for each group to share their mind maps with the larger group. Discuss similarities, differences, and key points brought forth by participants.

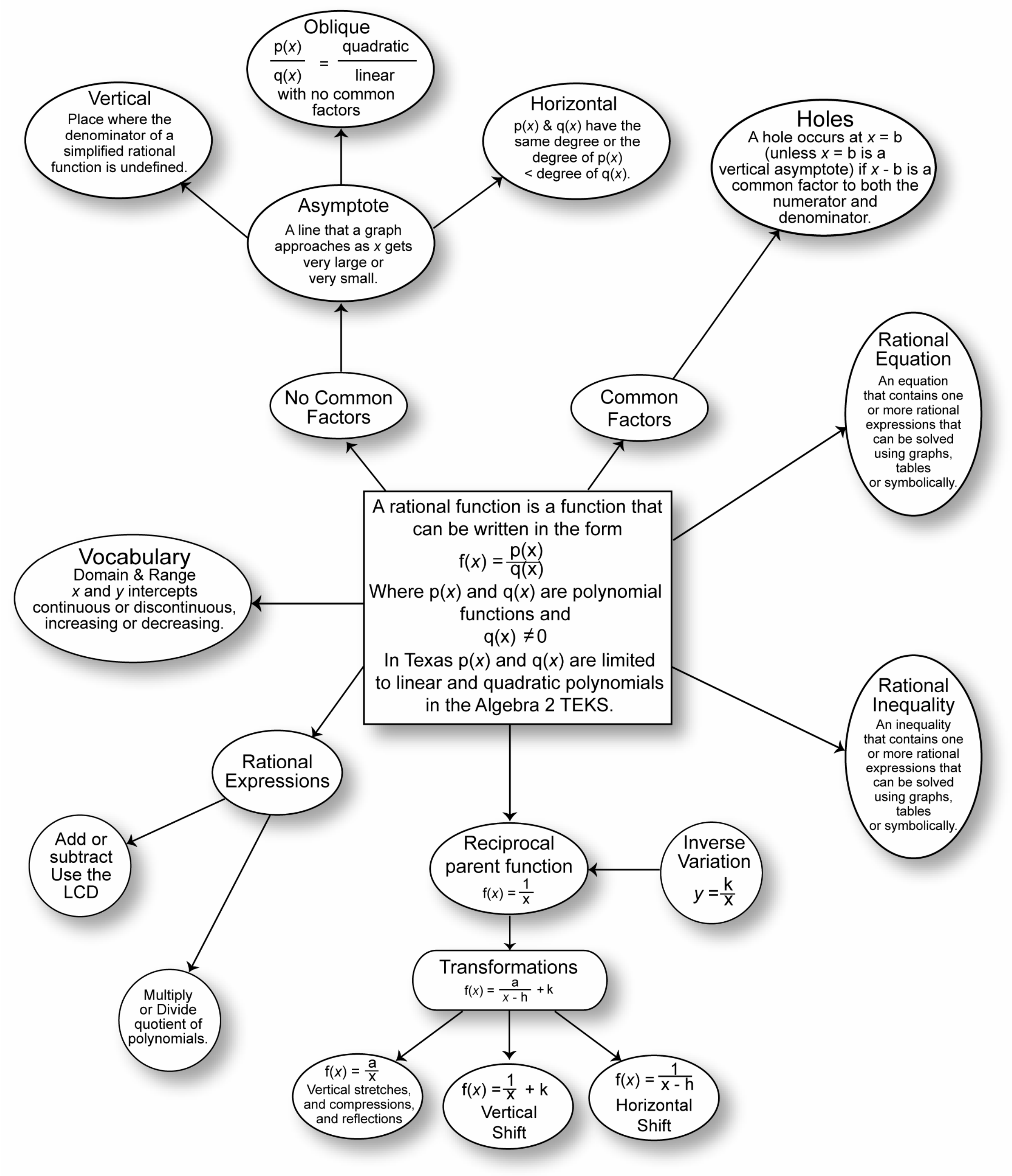
16. Distribute the vocabulary organizer template to each participant. Ask participants to construct a vocabulary model for the term rational functions.

17. When participants have completed their vocabulary models, ask participants to identify strategies from their experiences so far in the professional development that could be used to support students who typically struggle with Algebra II topics.

Note to Leader: *You may wish to have each small group brainstorm a few ideas first, then share their ideas with the large group while you record their responses on a transparency or chart paper.*

18. How would this lesson maximize student performance in Algebra II for teaching and learning the mathematical concepts and procedures associated with rational functions?
Responses may vary. Anchoring procedures within a conceptual framework helps students understand what they are doing so that they become more fluent with the procedures required to accomplish their tasks. Problems present themselves in a variety of representations; providing students with multiple procedures to solve a given problem empowers students to solve the problem more easily.

Rational Functions Mind Map



Vocabulary Organizer

Rational Functions

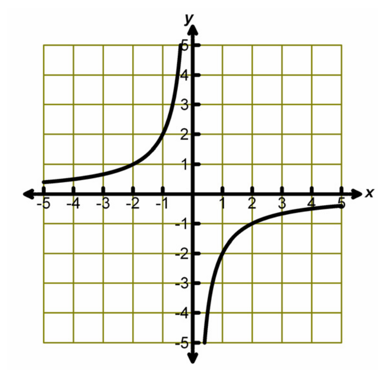
My definition

can be written in the form $\frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomials and q(x) ≠0

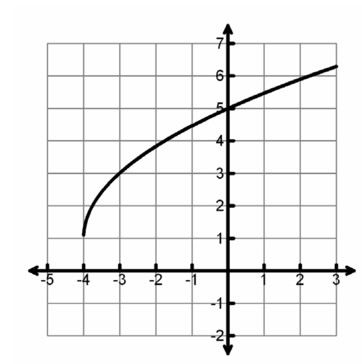
Personal Association

Similar to operations with fractions, may need a common denominator, may need to multiply by the reciprocal, the denominator cannot equal zero

Example



Non-Example



Elaborate

Leaders' Note: *In this phase, participants will extend their learning experiences to their classroom.*

1. **Distribute the 5E Student Lesson planning template. Ask participants to think back to their experences in the Explore phase. Pose the following task:**

   What might a student-ready 5E lesson on inverses of relations/functions look like?
   - **What would the Engage look like?**
   - **Which experiences/activities would students explore firsthand?**
   - **How would students formalize and generalize their learning?**
   - **What would the Elaborate look like?**
   - **How would we evaluate student understanding of inverses of relations/functions?**

2. **After participants have recorded their thoughts, direct them to the student lesson for inverses of relations/functions. Allow time for participants to review lessons.**

3. **How does this 5E lesson compare to your vision of a student-centered 5E lesson?**
   Responses may vary.

4. **How does this lesson help remove obstacles that typically keep students from being successful in Algebra II?**
   By connecting symbolic manipulation to conceptual understanding as revealed in other representations (such as graphing), students have other tools with which to solve meaningful problems.

5. **How does this lesson maximize your instructional time and effort in teaching Algebra II?**
   Taking time to create a solid conceptual foundation reduces the need for re-teaching time and effort.

6. **How does this lesson maximize student learning in Algebra II?**
   Using multiple representations and foundations for functions concepts allows students to make connections among different ideas. These connections allow students to apply their learning to new situations more quickly and readily.

7. **How does this lesson accelerate student learning and increase the efficiency of learning?**
   Foundations for functions concepts such as function transformations transcend all kinds of functions. A basic toolkit for students to use when working with functions allows students to rethink what they know about linear and quadratic functions while they are learning concepts and procedures associated with other function families.

8. **Read through the suggested strategies on Strategies that Support English Language Learners. Consider the possible strategies designed to increase the achievement of English language learners.**

As participants read through the strategies that support English language learners and strategies that support students with special needs, they may notice that eight of the ten strategies are the same. The intention is to underscore effective teaching practices for all students. However, English language learners have needs specific to language that students with special needs may or may not have. The two strategies that are unique to the English language learners reflect an emphasis on language. Students with special needs may have prescribed modifications and accommodations that address materials and feedback. Students with special needs often benefit from progress monitoring with direct feedback and adaptation of materials for structure and/or pacing. A system of quick response is an intentional plan to gather data about a student's progress to determine whether or not the modification and (or) accommodation are (is) having the desired effect. The intention of the strategies is to provide access to rigorous mathematics and support students as they learn rigorous mathematics.

9. **What evidence of these strategies do you find in this portion of the professional development?**
   Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening. This contributes to the teacher's ability to create an emotionally safe environment for learning.*

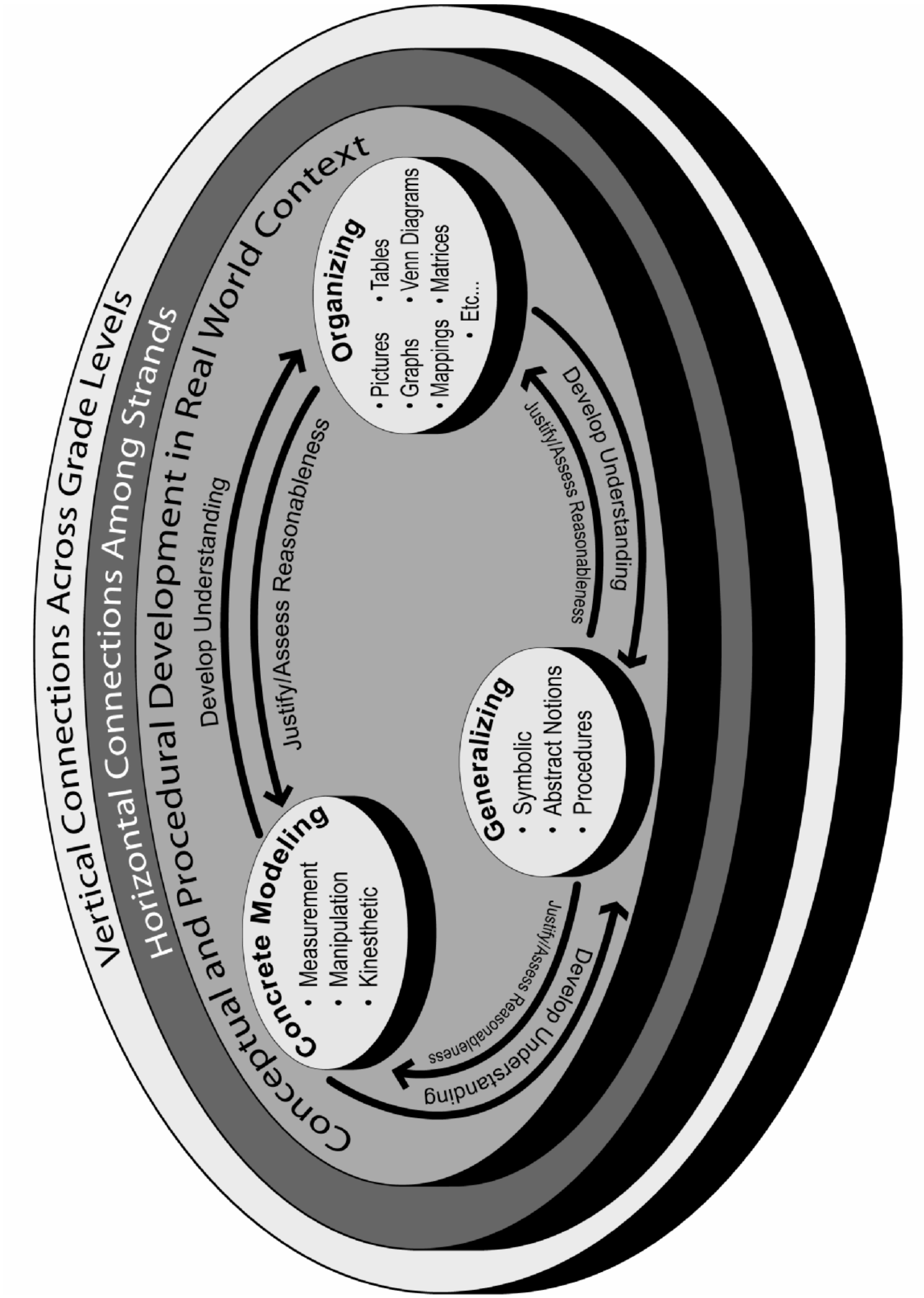
10. **Which strategies require adaptation of the materials in this portion of the professional development?**
    Responses may vary.

11. **Read through the suggested strategies on Strategies that Support Students with Special Needs. Consider the possible strategies designed to increase the achievement of students with special needs.**

12. **What evidence of these strategies do you find in this portion of the professional development?**
    Responses may vary. ***Note:*** *Some strategies reflect teacher behaviors. The presenter may need to prompt participants to consider how the professional development materials support the needed teacher behaviors. For example, a student lesson may outline a structured approach for exploration so that the activity is non-threatening. This contributes to the teacher's ability to create an emotionally safe environment for learning.*

13. **Which strategies require adaptation of the materials in this portion of the professional development?**
    Responses may vary.

Processing Framework Model



Vocabulary Organizer

My definition

Personal Association

Example

Non-Example

5E Student Lesson Planning Template

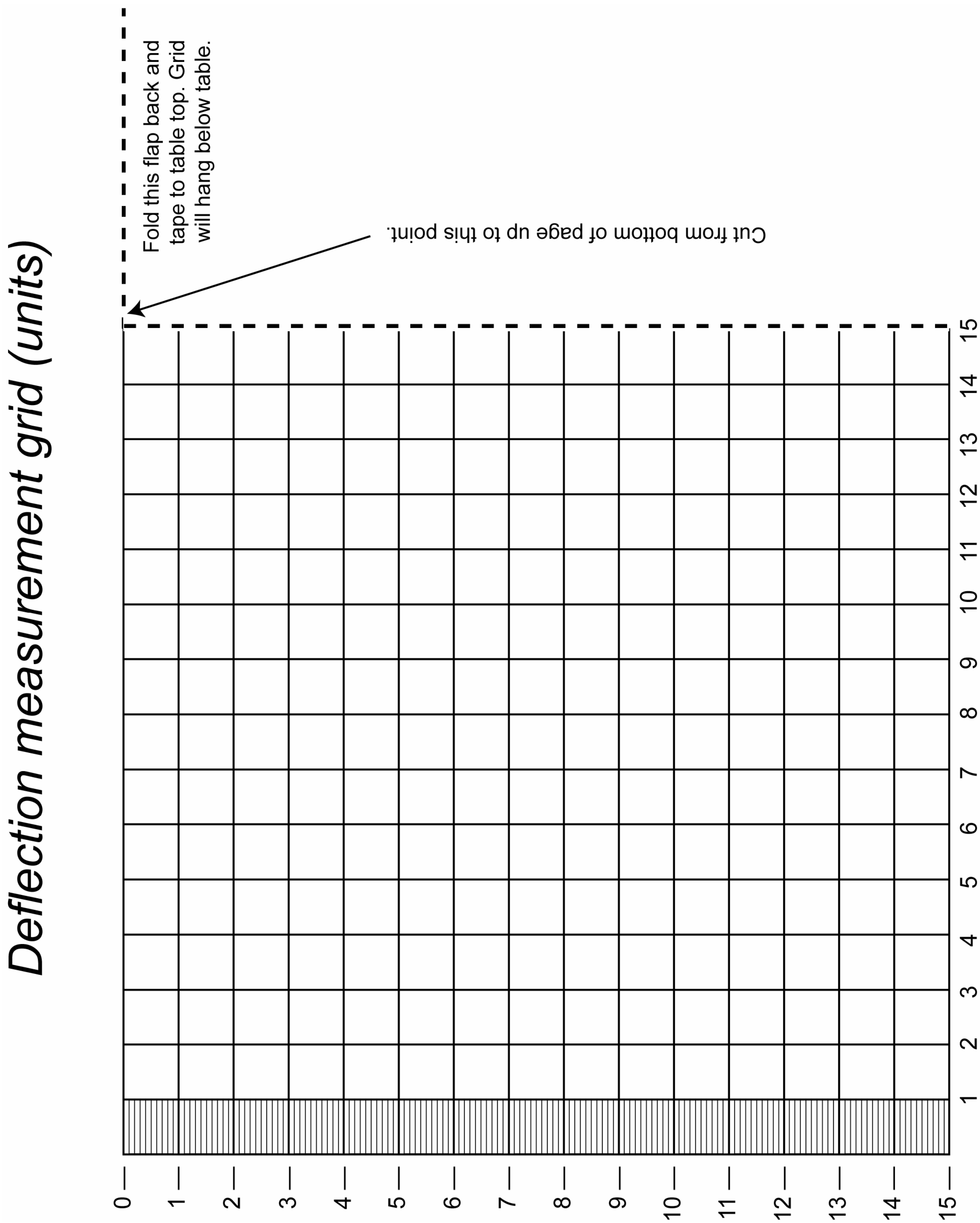
Description	Activity
Engage The activity should be designed to generate student interest in a problem situation and to make connections to prior knowledge. The instructor initiates this stage by asking meaningful questions, posing a problem to be solved, or by showing something intriguing.	
Explore The activity should provide students with an opportunity to become actively involved with the key concepts of the lesson through a guided exploration requiring them to probe, inquire, and question. The instructor actively monitors students as they interact with each other and the activity.	
Explain Students collaboratively begin to sequence events/facts from the investigation and communicate these findings to each other and the instructor. The instructor, acting in a facilitation role, formalizes student findings by providing further explanations and additional meaning or information, such as correct terminology.	
Elaborate Students extend, expand, or apply what they have learned in the first three stages and connect this knowledge with prior learning to deepen understanding. Instructors can use the Elaborate stage to verify students' understandings.	
Evaluate Evaluation occurs throughout students' learning experiences. More formal evaluation can be conducted at this stage. Instructors can determine whether the learner has reached the desired level of understanding the key ideas and concepts.	

Strategies that Support English Language Learners (ELL)

Strategy	Explore, Explain, Elaborate 4
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond.	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Be conscious of tone and diction. Speak slowly and distinctly.	
Incorporate language skills (reading, writing, speaking, and listening) into instruction.	

Strategies that Support Students with Special Needs

Strategy	Explore, Explain, Elaborate 4
Create an emotionally safe environment for learning.	
Provide ample time for students to process questions before expecting them to respond.	
Encourage students to work together in structured activity.	
Link learning to prior knowledge.	
Teach conceptual vocabulary with organizers, concept mappings, and word walls.	
Use a variety of instructional techniques including manipulatives and multiple representations.	
Use scaffolding techniques to build from simple understandings to complex understandings, making conceptual and procedural connections explicit.	
Prompt student to compare and contrast concepts, procedures, and generalizations.	
Use a system of quick response to needs and accommodations including progress monitoring to inform instruction.	
Accommodate materials for format, structure, sequence, etc. as needed.	


Deflection measurement grid (units)
Fold this flap back and tape to table top. Grid will hang below table.
Cut from bottom of page up to this point.
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Transparency

Average Amount of Deflection in the Linguine Bundles

Number of Pieces of Linguine in the Bundle	Amount of Deflection for Each Team												Average
	A	B	C	D	E	F	G	H	I	J	K	L	
1													
2													
3													
4													
5													
6													
7													
8													

Activity Page

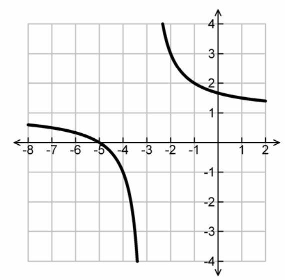
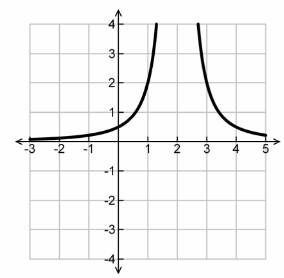
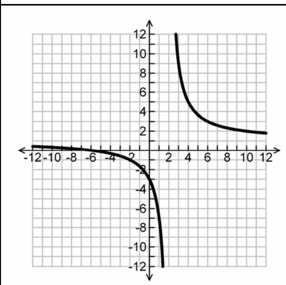
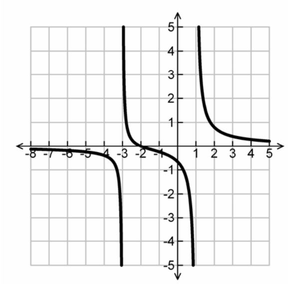
Rational Function Card Sort

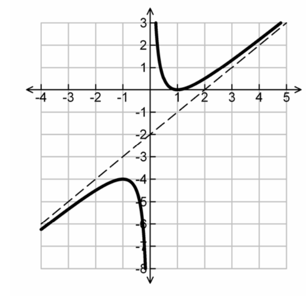
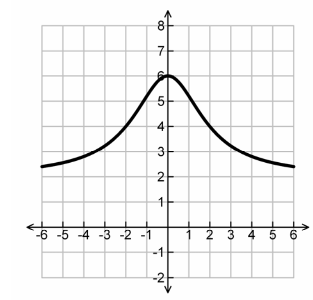
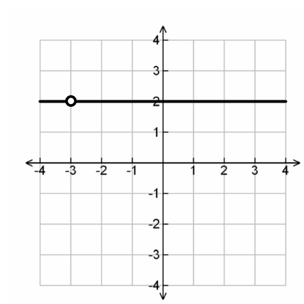
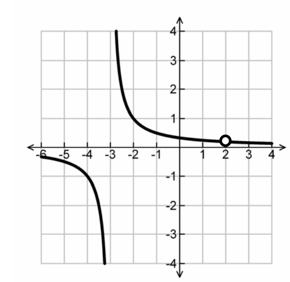
$b(x) = x$	$c(x) = e^x$
$d(x) = 6 + \dfrac{10}{x-2}$	$f(x) = 7 - 3\sqrt{x-2}$
$g(x) = 3^{x-4} + 10$	$h(x) = \dfrac{x^2 + 6x - 3}{x+4}$
$j(x) = -\dfrac{1}{4}$	$k(x) = \sqrt{2}x$
$m(x) = \log(2x+2) - 10$	$n(x) = \dfrac{2x+3}{3x-1}$

$p(x)=\sqrt{3}x^2+2x-7$	$q(x)=2x^2+3$
$r(x)=\sqrt{2x+1}$	$s(x)=10^{2x}+4$
$t(x)=\frac{2}{3}x^2-\frac{3}{4}$	$u(x)=\frac{\sqrt{x+2}}{\sqrt{2x-1}}$
$v(x)=\ln\left(\ln 2x+\frac{2}{3}\right)-\frac{3}{4}$	$w(x)=\frac{\sqrt{2}x^2+4}{\sqrt{10}x^2-4x+\sqrt{15}}$
$y(x)=5\frac{1}{2}$	$z(x)=(3+i)x^2+2ix$

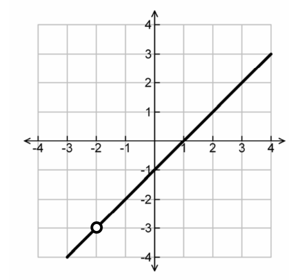
Rational Function Card Match

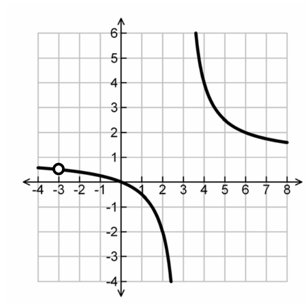
Activity Cards

$f(x)=\dfrac{x+5}{x+3}$ **A**	$f(x)=\dfrac{2}{x+3}+1$	$x=-3$ $y=1$ x-intercept at $(-5,\ 0)$ y-intercept at $(0,\ \frac{5}{3})$	
$f(x)=\dfrac{2}{x^2-4x+4}$ **B**	$f(x)=\dfrac{2}{(x-2)^2}$	$x=2$ $y=0$ no x-intercepts y-intercept at $(0,\ 0.5)$	
$f(x)=\dfrac{x+6}{x-2}$ **C**	$f(x)=\dfrac{8}{x-2}+1$	$x=2$ $y=1$ x-intercept at $(-6,\ 0)$ y-intercept at $(0,\ -3)$	
$f(x)=\dfrac{x+2}{x^2+2x-3}$ **D**	$f(x)=\dfrac{x+2}{(x+3)(x-1)}$	$x=1$ and $x=-3$ $y = 0$ x-intercept at $(-2,\ 0)$ y-intercept at $(0,\ -\frac{2}{3})$	

$f(x)=\dfrac{x^2-2x+1}{x}$ **E**	$f(x)=x-2+\dfrac{1}{x}$	$y=x-2$ $x=0$ x-intercept at (1, 0) no y-intercepts	
$f(x)=\dfrac{2x^2+24}{x^2+4}$ **F**	$f(x)=\dfrac{16}{x^2+4}+2$	$y=2$ no vertical asymptotes no x-intercepts y-intercept at (0, 6)	
$f(x)=\dfrac{2x+6}{x+3}$ **G**	$f(x)=2$, $x\neq -3$	no horizontal asymptote removable discontinuity (hole) at (−3, 2) no x-intercepts y-intercept at (0, 2)	
$f(x)=\dfrac{x-2}{x^2+x-6}$ **H**	$f(x)=\dfrac{1}{x+3}$, $x\neq -2$	x = –3 and y = 0 removable discontinuity (hole) at (2, 0.2) no x-intercepts y-intercept at (0, $\frac{1}{3}$)	

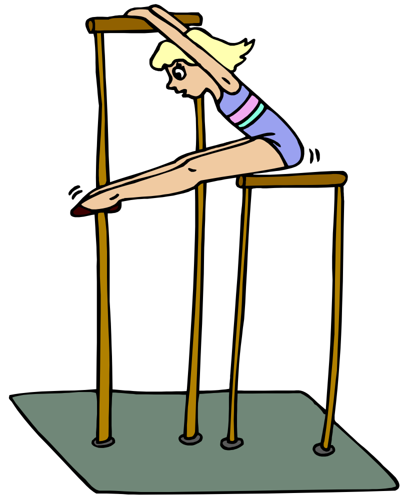
<table>
<tr><td rowspan="4">$f(x)=\dfrac{x^2+x-2}{x+2}$

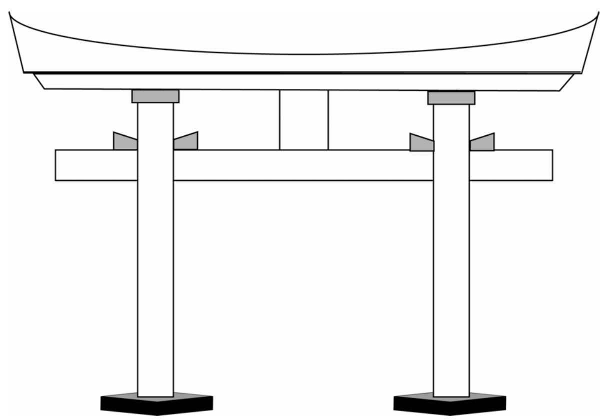
I</td><td rowspan="4">$f(x)=x-1,$
$x\neq -2$</td><td>no horizontal asymptote</td><td rowspan="4"></td></tr>
<tr><td>removable discontinuity (hole) at (−2, −3)</td></tr>
<tr><td>*x*-intercept at (1, 0)</td></tr>
<tr><td>*y*-intercept at (0, −1)</td></tr>
<tr><td rowspan="4">$f(x)=\dfrac{x^2+3x}{x^2-9}$

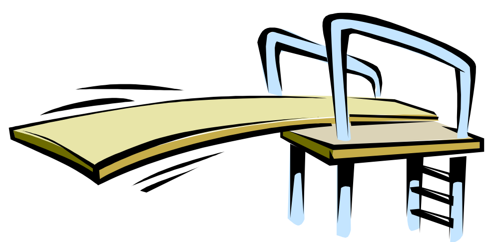
J</td><td rowspan="4">$f(x)=\dfrac{3}{x-3}+1,$
$x\neq -3$</td><td>x = 3 and $y=1$</td><td rowspan="4"></td></tr>
<tr><td>removable discontinuity (hole) at (−3, 0.5)</td></tr>
<tr><td>*x*-intercept at (0, 0)</td></tr>
<tr><td>*y*-intercept at (0, 0)</td></tr>
</table>

Participant Pages: Rational Functions

How are the objects below alike? How are they different?







<u>Part 1</u>: Linguine Cantilever

A cantilever is a projecting structure that is secured at only one end and carries a load on the other end. Diving boards and airplane wings are examples of horizontal cantilevers. Flagpoles and chimneys are vertical cantilevers. One of the most famous examples of a cantilever in architecture, which is shown below, is the Frank Lloyd Wright designed home, Fallingwater. The strength of a cantilever can be affected by variables such as length, load, cross sectional area, temperature, or elasticity. In this activity, you will be investigating the relationship between the thickness of a cantilever and the deflection in the cantilever when weight is added at the end.

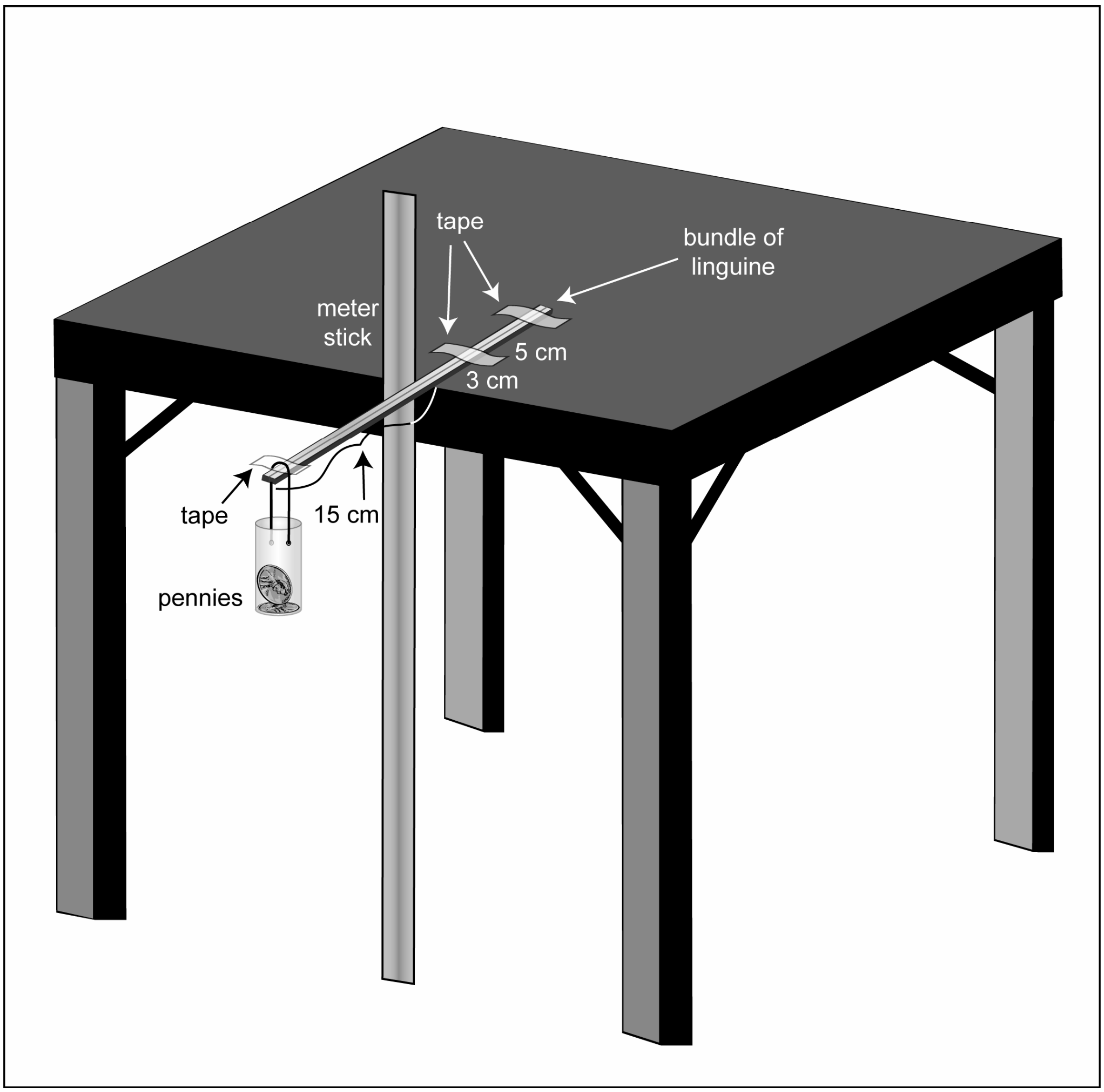
In this investigation, you will keep the length of a piece of linguine that is hanging over the edge of a desk constant as you collect data on how much the linguine deflects. Deflection is the amount that the linguine bends in the downward direction. The number of pieces of linguine will change. Since you want to keep all variables (except for the ones you are investigating) constant make sure to pay attention to the hints listed with the instructions.

Have participants get into groups of three. Each person in the group has a job.

Materials manager: Get the necessary materials, direct the team in setting up the investigation

Measures manager: Measure the amount of deflection as the investigation proceeds

Data manager: Record the necessary measurements in the table, share the data with the team



Data Collection Set-up Instructions

Step 1. The materials person should get the necessary materials and begin to make bundles of linguine. Each bundle of 1, 2, 3, 4, 5, 6, 7, and 8 pieces of linguine should be taped one inch from each end. Since linguine is not all exactly the same length, try to keep one end of the bundle lined up.

Step 2. Tape a short piece of string to the 35mm film canister to form a handle. If you do not have a film canister, use a baggie, transparent tape, and a paper clip to build a weight to hang from the linguine.

Step 3. Tape one piece of linguine with 15 centimeters hanging over the edge of a desk. Put one piece of tape approximately 3 cm from the edge of the table. Place a second piece of tape over the end of linguine. Place the load (film canister) on the end of the linguine that is hanging over the edge of the desk. Slowly place pennies in the film canister until the linguine breaks. Wait 15 seconds before adding an additional penny. Use one less penny than the number required to break one piece of linguine as the load in your bucket for the remainder of this data collection experiment.

Step 4. Tape a meter stick perpendicularly to the floor next to a desk.

Step 5. Measure the linguine's height above the floor without the film canister attached. (Hint: It is easier to consistently measure the height using the bottom of the linguine.)

Step 6. Place your pennies into the bucket. (Hint: Place the pennies gently, throwing pennies into the bucket will alter the results.)

Step 7. Place the bucket on the end of the linguine that is hanging over the edge of the desk. (Hint: Place the string at the same point on the linguine for each trial. Use a piece of masking tape to hold the bucket onto the linguine.)

Step 8. Wait 15 seconds. Measure the amount of deflection in the linguine. (Hint: The easiest method for measuring deflection is to use the eraser end of a pencil to line up the deflection of the end of the linguine with its measure on the meter stick.) Record your measurements in the table.

Step 9. Repeat the procedure with two pieces of linguine taped together still hanging 15 centimeters over the edge of the desk. Measure the deflection of the bundle of linguine.

Step 10. Continue repeating the procedure with additional pieces of linguine until you measure deflection with eight pieces taped together. Continue to record your data.

1. Fill in the table with the data you collected.

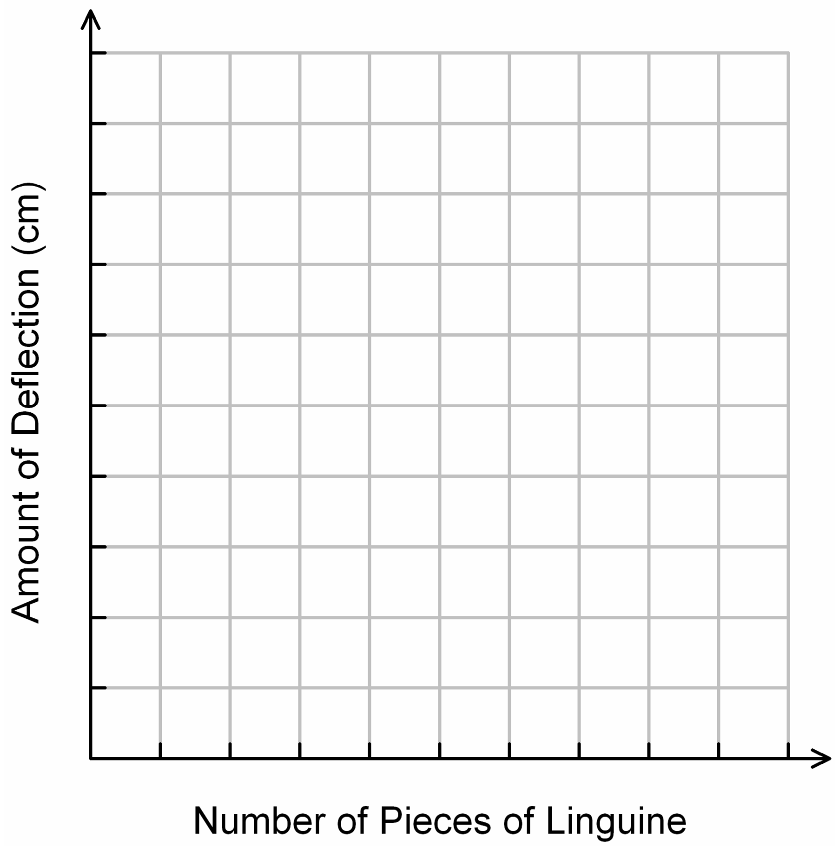
Number of Pieces of Linguine in the Bundle (x)	Starting Height of Linguine Bundle Above the Floor	Height of Linguine Bundle Above the Floor After the Load is Placed	Amount of Deflection in the Linguine (y)	Product of x and y ($x \cdot y$)
1				
2				
3				
4				
5				
6				
7				
8				

2. Write a dependency statement relating the two variables.

3. What is a reasonable domain for the set of data?

4. What is a reasonable range for the set of data?

5. Make a scatterplot of the data you collected.



6. Verbally describe what happens in this data collection investigation.

7. Is this data set continuous or discrete? Why?

8. Does the set of data represent a function? Why?

9. Does the data appear to be a linear, quadratic, exponential or some other type of parent function? Why do you think so?

10. Is the function increasing or decreasing?

11. Is the rate of change constant for this set of data?

12. Determine a function rule that models the set of data you collected.

13. To get a better model, add your set of data to the data of the entire group. Each group should send their data manager to the overhead to fill in the data collected for their group. Record the additional data in the table below. Find the average deflection for each bundle of linguine for the entire group.

Number of Pieces of Linguine in the Bundle	Amount of Deflection for Each Team												Average
	A	B	C	D	E	F	G	H	I	J	K	L	
1													
2													
3													
4													
5													
6													
7													
8													

14. Using the entire group's data, what function would you now use to model this situation?

15. How does this investigation connect to the TEKS from previous courses?

16. What are the key points students need to understand about the Linguine Cantilever before continuing the investigation of rational functions?

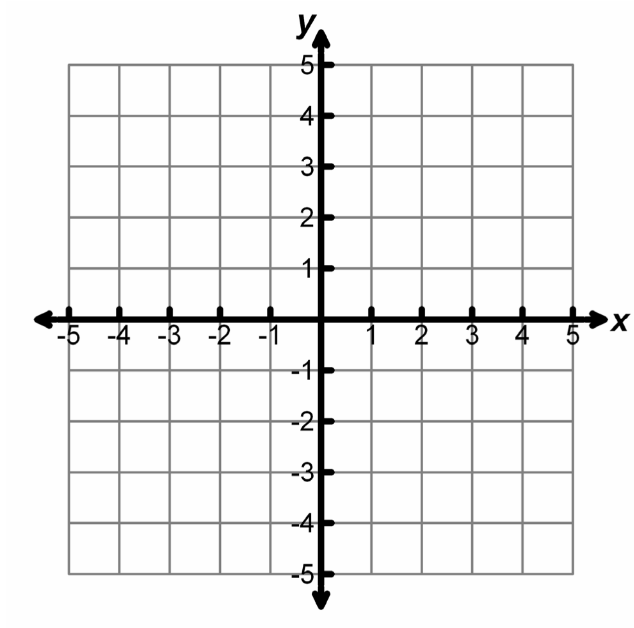
<u>Part 2</u>. Transformations to $f(x)=\dfrac{1}{x}$

1. What is the reciprocal of the linear parent function, $f(x)=x$?

2. Let's investigate some of the attributes of the function and its reciprocal. Fill in the tables with several values for each function. Draw a sketch of the graphs of the two functions on the same set of axes.

$f(x)=x$	
x	***y***
–3	
–2	
–1	
–0.5	
–0.1	
0	
0.1	
0.5	
1	
2	
3	

$f(x)=\dfrac{1}{x}$	
x	***y***
–3	
–2	
–1	
–0.5	
–0.1	
0	
0.1	
0.5	
1	
2	
3	



3. Using your graphing calculator (if necessary), fill in the tables below. Let $f(x)=x$ be Y_1, and let $g(x)=\dfrac{1}{x}$ be Y_2.

$$Y_1=x$$
$$Y_2=\frac{1}{x}$$

$Y_1=x$		$Y_2=\dfrac{1}{x}$
	Intervals where the function is increasing	
	Intervals where the function is decreasing	
	Intervals where the function is undefined	
	Coordinates of the *x*-intercepts (zeros)	
	Equations of any asymptotes	

4. What do you notice about the graphs of the linear parent function and its reciprocal?

5. Where do the linear parent function and its reciprocal intersect?

6. How could you have your students investigate what happens to $f(x)$ as x gets closer and closer to 0 using the graphing calculator?

7. How could you have your students investigate what happens to $f(x)$ as x gets larger and larger?

8. How do the Algebra II TEKS name this new parent function?

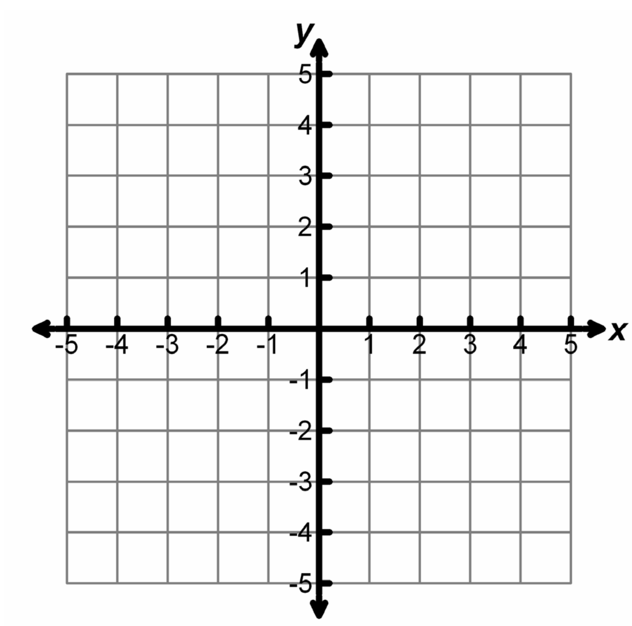
9. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x)=\frac{1}{x}$, when it is multiplied by a constant as in the examples below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results.

$Y_1=\frac{1}{x}$

$Y_2=\frac{3}{x}$

$Y_3=\frac{0.1}{x}$

x	$Y_1=\frac{1}{x}$	$Y_2=\frac{3}{x}$	$Y_3=\frac{0.1}{x}$
−3			
−2			
−1			
0			
1			
2			
3			
4			



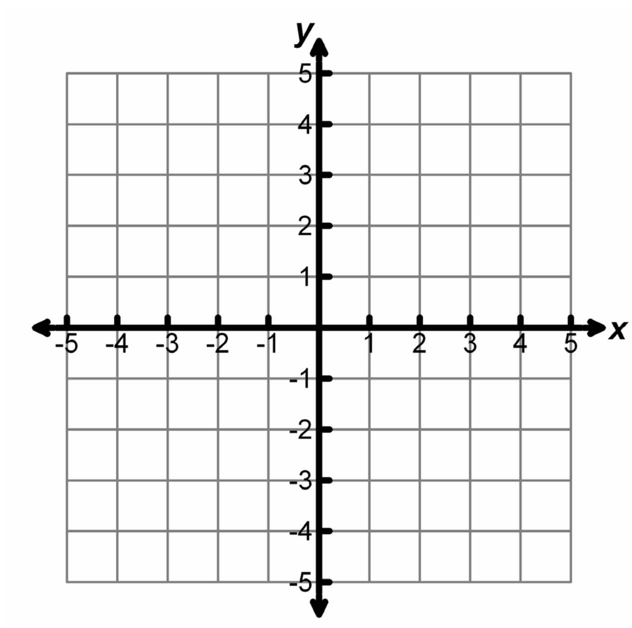
10. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x) = \frac{1}{x}$, when it is multiplied by a negative constant as in the examples below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results which includes any asymptotes.

$Y_1 = \frac{1}{x}$

$Y_2 = -\frac{1}{x}$

$Y_3 = -\frac{4}{x}$

x	$Y_1 = \frac{1}{x}$	$Y_2 = -\frac{1}{x}$	$Y_3 = -\frac{4}{x}$
-3			
-2			
-1			
0			
1			
2			
3			
4			



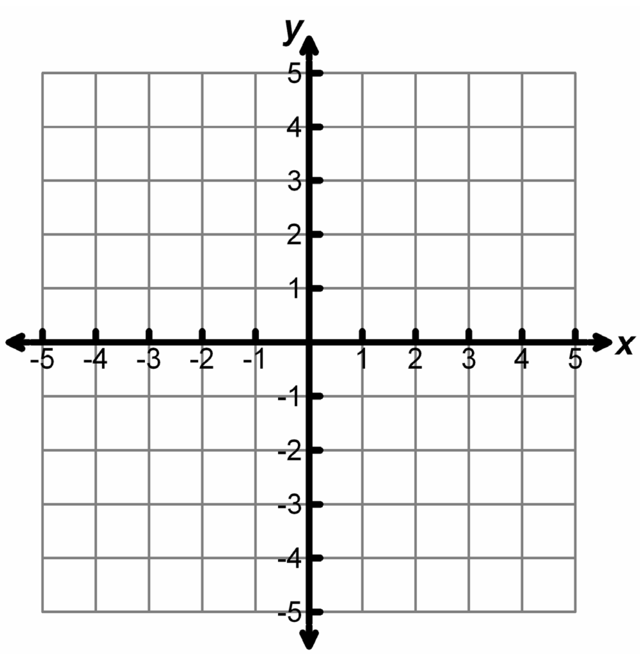
11. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x)=\frac{1}{x}$, if a constant is added to the function, as in the three functions listed below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results which includes any asymptotes.

$$Y_1=\frac{1}{x}$$

$$Y_2=\frac{1}{x}+3$$

$$Y_3=\frac{1}{x}-2$$

x	$Y_1=\frac{1}{x}$	$Y_2=\frac{1}{x}+3$	$Y_3=\frac{1}{x}-2$
–3			
–2			
−1			
0			
1			
2			
3			
4			



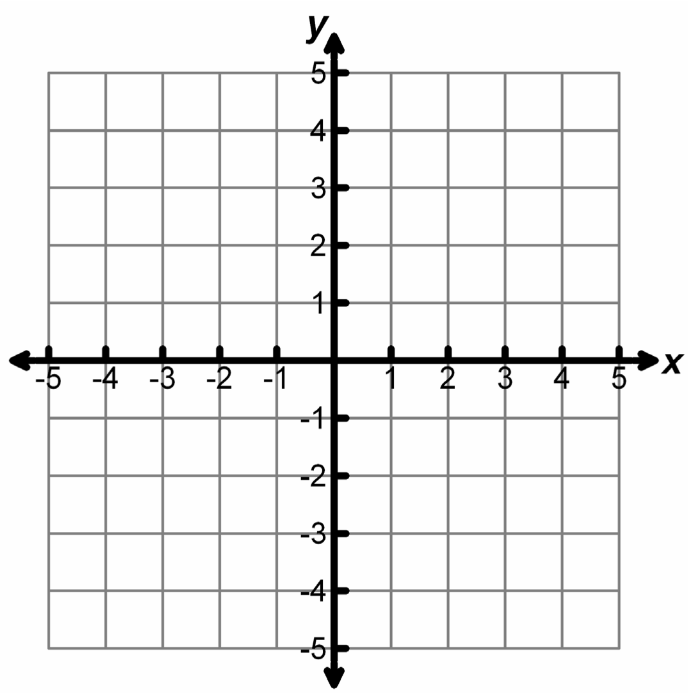
12. Using your graphing calculator, describe what happens to the reciprocal parent function, $g(x)=\frac{1}{x}$, if a constant is added to the x-coordinate in the denominator, as in the three functions listed below. List a few values from the table feature of the graphing calculator in the table below. Show how the transformation is evidenced in your table. Draw a sketch to aid in the description of your results which includes any asymptotes.

$Y_1=\frac{1}{x}$

$Y_2=\frac{1}{x+3}$

$Y_3=\frac{1}{x-2}$

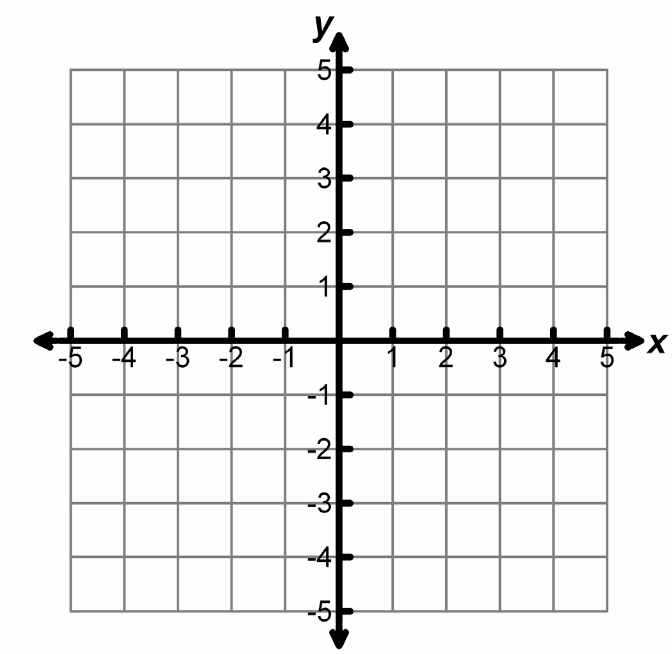
x	$Y_1=\frac{1}{x}$	$Y_2=\frac{1}{x+3}$	$Y_3=\frac{1}{x-2}$
-3			
-2			
-1			
0			
1			
2			
3			
4			



13. Predict, describe and then sketch the transformations to the reciprocal parent function in the function below.

$$f(x) = -\frac{2}{x+1} - 3$$

x	$Y_1 = \frac{1}{x}$	$Y_1 = -\frac{2}{x}$	$Y_1 = -\frac{2}{x+1}$	$Y_2 = -\frac{2}{x+1} - 3$
−3				
−2				
−1				
0				
1				
2				
3				
4				



14. Describe how transformations to the reciprocal function are similar to transformations to other parent functions.

15. What are the key points students need to understand about transformations to the reciprocal function before continuing the investigation of rational functions?

<u>Part 3</u>: Card Sort and Match

1. Sort the first deck of cards into two (and only two) groups. Describe your method for sorting.

2. Using the cards match the rational form of each function to the transformation form or factored form, the equations of any asymptotes, the coordinates of any removable discontinuities (holes), and the graph of the function. Record your answers in the table.

Card Match

Rational Function Form	Transformation Form	Discontinuities, Asymptotes, Other Noteworthy Points	Graph
A			
B			
C			

D			
E			
F			
G			

H			
I			
J			

3. What are the key points students need to understand about rational functions to be able to do the Rational Function Card Match?

<u>Part 4</u>: Length of a Yellow Light

One of the formulas traffic engineers use to help them calculate the length of time a traffic light should remain yellow is

$$Y(t) = t + \frac{v}{2a} + \frac{w+L}{v}.$$

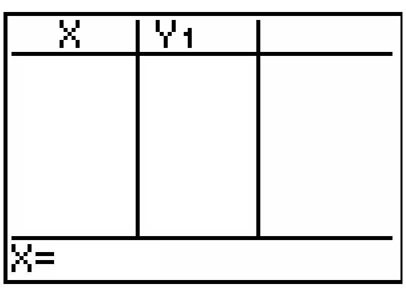
The formula takes into account reaction time, braking time, and intersection clearance time. The variables used in this calculation are:

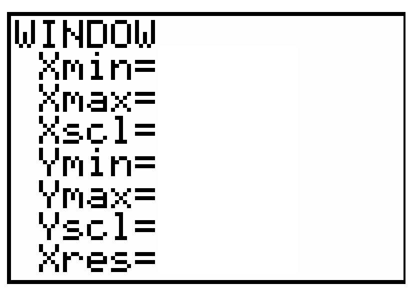
- t = reaction time (usually 1 second)
- v = velocity of the vehicle (in feet/second)
- a = deceleration rate (approximately 10 feet/ sec^2)
- w = width of the intersection (feet)
- L = length of the vehicle (feet)

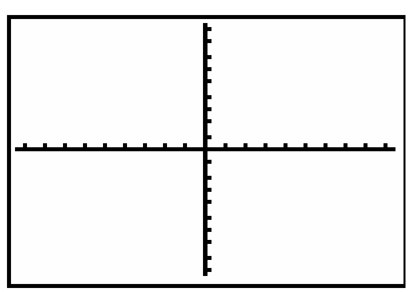
In order to calculate the speed limit for a certain intersection that is 48 feet wide, the engineer uses an average car length of 18 feet. She can calculate the length of time the traffic signal should remain yellow at that intersection based on the velocity in feet/second using the following formula:

$$Y(t) = 1 + \frac{v}{20} + \frac{66}{v}.$$

1. If the posted speed limit at the intersection is 55 miles per hour, how long should the signal remain yellow?

<table>
<tr>
<td>Solution using a table:


</td>
<td>Solution using a graph:



</td>
<td>Symbolic solution:</td>
</tr>
</table>

2. For a traffic signal to remain yellow for 4 seconds what should the department of transportation post as the speed limit?

3. If the speed of vehicles at a particular intersection varies between 30 and 50 mph, how long do you think the traffic signal should remain yellow?

4. A tractor trailer that is approximately 36 feet long travels through the same intersection when the signal remains yellow for 6 seconds. Based on the formula, how fast should the tractor trailer be allowed to drive through the intersection? How fast should a car be allowed to drive through the intersection?

$$Y(v) = 1 + \frac{v}{20} + \frac{84}{v}$$

5. Do you think this is a good model for length of time that traffic signals should remain yellow for every *x* value in the domain?

Bonus question:

6. What is the oblique asymptote for this rational function? Graph both the function and the asymptote on your graphing calculator.

$$Y(v) = 1 + \frac{v}{20} + \frac{66}{v}$$

7. Do you think that students should solve every problem involving rational functions symbolically? What understanding would students gain from solving with tables and graphs?

Evaluate

Leader Notes: Reflecting and Extending

Purpose:
The Evaluate phase will assess participants' understanding of the instructional components of the professional development that address potential stumbling blocks for teaching and learning. This phase also allows participants to reflect upon the learning resulting from the professional development.

Descriptor:
Participants will view video excerpts of a student lesson from this professional development. They will note effective teaching behaviors and evidences of student learning. Participants will use their observations about the video excerpts and their learning during the professional development to discuss the implications for personal practice.

Duration:
45 minutes

TEKS:

a5	Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.
a6	Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.
2A.10	**Rational functions.** The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.
2A.10A	The student is expected to use quotients of polynomials to describe the graphs of rational functions, predict the effects of parameter changes, describe limitations on the domains and ranges, and examine asymptotic behavior.
2A.10B	The student is expected to analyze various representations of rational functions with respect to problem situations.

2A.10C The student is expected to determine the reasonable domain and range values of rational functions, as well as interpret and determine the reasonableness of solutions to rational equations and inequalities.

2A.10G The student is expected to use functions to model and make predictions in problem situations involving direct and inverse variation.

TAKS™ Objectives:
While the Algebra II TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.
- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance:	Projection system for video
Presenter Materials:	DVD with video excerpts of the 5E Instructional Model
Per participant:	**Watch This!**, **Debrief This!**, and **What Next?** activity pages

Leader Notes:
The purpose of this activity is to evaluate participants' growth in understanding related to the professional development. Participants should be asked to justify their statements and apply knowledge acquired in earlier phases of the training.

Evaluate

Part 1 – Reflecting on Teaching and Learning through Video Analysis (30 minutes)
1. Distribute the **Watch This!** activity page to each participant.

2. Play the excerpt from the Engage phase of the student lesson. When the screen for the Explore phase pops up pause the video for two minutes and allow participants to note observations about what the teacher is doing, what the students are doing, and what evidences of learning are heard or seen.

3. Repeat this process for each phase of the 5E Instructional Model.

4. Distribute the **Debrief This!** activity pages to each participant.

5. After the groups have answered their questions, prompt each group to share their observations and the implications of their observations with the whole group.

Debrief This!

1. **How does the teacher set the stage for the lesson?**
   Responses may vary. She shows slides of different types of cantilevers. She includes the Frank Lloyd Wright home, Fallingwater.

2. **How does the teacher relate new contexts to students' prior knowledge?**
   Responses may vary. She connects the word cantilever to a diving board.

3. **How does the teacher place the responsibility on the students for the lesson?**
   Responses may vary. Each person in a group has a responsibility during the data collection.

4. **What is the teacher doing that you might not normally see in an Algebra II classroom?**
   Responses may vary, such as facilitating rather than lecturing, asking questions, and introducing new vocabulary.

5. **What student behavior do you see in the Explore that you might not normally see in an Algebra II classroom?**
   Responses may vary, such as: working together, talking to each other, discussing mathematics, posting work on chart paper, and collecting data.

6. **How is the Evaluate different from what you might normally see in an Algebra II classroom?**
   Responses may vary, such as: card match as opposed to homework problems, students working together and discussing mathematics.

7. **How does student behavior change as the 5E lesson progresses? Why do you think it changes?**
   Responses may vary. Students become more talkative and appear more engaged. The lesson is presented so that it is related to a real world context with students collecting data and modeling their data with a function. Students discover transformations to the reciprocal parent function using their calculators as a manipulative.

Part 2 – Extending to Personal Practice (15 minutes)

1. Distribute the **What Next?** activity page to each participant.

2. Prompt participants to respond to the questions posed on the **What Next?** activity page.

3. Prompt participants to share one response within their table groups.

4. After each person has shared a response, prompt the group to determine a summary statement about their discussion.

5. Prompt each group to share its summary statement to draw closure to the professional development.

Watch This!

5E Instructional Model Phases	The teacher…	The student…	Evidence of learning…
Engage			
Explore			
Explain			
Elaborate			
Evaluate			

Debrief This!

1. How does the teacher set the stage for the lesson?

2. How does the teacher relate new contexts to students' prior knowledge?

3. How does the teacher place the responsibility on the students for the lesson?

4. What is the teacher doing that you might not normally see in an Algebra II classroom?

5. What student behavior do you see in the Explore that you might not normally see in an Algebra II classroom?

6. How is the Evaluate different from what you might normally see in an Algebra II classroom?

7. How does student behavior change as the 5E lesson progresses? Why do you think it changes?

What Next?

1. What topics comprise the next unit of instruction for your students?

2. What topics within this unit present challenges for your students as they learn? Why?

3. What opportunities for modeling exist in these challenging topics?

4. What opportunities for organizing exist in these challenging topics?

5. What opportunities for generalizing exist in these challenging topics?

6. What five questions do you want your students to be able to answer related to modeling, organizing, and generalizing within this unit?

7. How might facilitating connections among the processes of modeling, organizing, and generalizing help your students be more successful in this unit of study?

Student Lesson: Inverses of Functions

TEKS:

2A.1 **Foundations for functions.** The student uses properties and attributes of functions and applies functions to problem situations.

2A.1A The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations;

2A.1B The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

2A.2 **Foundations for functions.** The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

2A.2A The student is expected to use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

2A.4 **Algebra and geometry.** The student connects algebraic and geometric representations of functions.

2A.4A The student is expected to identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

2A.4B The student is expected to extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

2A.4C The student is expected to describe and analyze the relationship between a function and its inverse.

Objectives:

At the end of this student lesson, students will be able to:

- find the inverse of a given function;
- describe the inverse relations and functions using graphs, tables, and algebraic methods;

TAKS™ Objectives Supported:

While the Algebra 2 TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials

Prepare in Advance:

- Transparency 1 and Transparency 2
- Student copies of **Know When to Hold 'Em, Know When to Fold 'Em, Multiple Representations of Inverses** and **How Cheesy.**

For each student group of 3 - 4 students:

- Several sheets of patty paper

For each student:

- Graphing calculator
- 1 copy per student of **Rising Water**
- 1 copy per student of **Know When to Hold 'Em**
- 1 copy per student of **Know When to Fold 'Em**
- 1 copy per student of **Multiple Representations of Inverses**
- 1 copy per student of **How Cheesy**

Engage

The Engage portion of the lesson is designed to provide students with a concrete introduction to the concept of inverses. This part of the lesson is designed for whole group instruction and input.

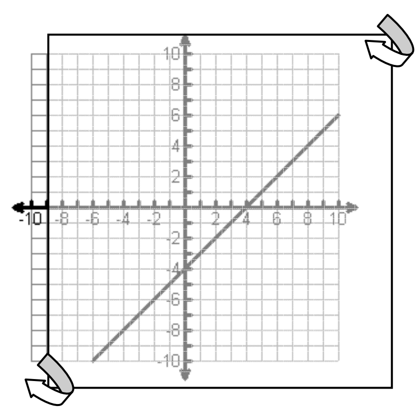
At the end of the Engage phase, students should be able to identify which transformations yield a functional relationship and which yield a non-functional relationship.

1. Distribute one copy of **Rising Water** to each student and post the **Transparency: Vince Bayou**. Pose the question on the activity sheet to students and allow them a few minutes to record their responses.

2. Distribute patty paper and one copy of the **Know When to Hold 'Em** activity sheet to each student**.**

3. Prompt students to trace each graph onto a sheet of patty paper. Also prompt them to trace and label the axes on their patty paper.

4. Demonstrate for students taking the top-right corner of the patty paper in your right hand and the bottom-left corner in your left hand and flipping the patty paper over so that the top-left corner and the bottom-right corner of the patty paper change places (see figure at right).



5. Prompt students to align the origin on the patty paper with the origin of their original graph and align the axes on the patty paper with the axes on the graph. While observing the original position and the new position of the graph they should answer the related questions.

6. Facilitate the activity using the Facilitation Questions.

Facilitation Questions – Engage Phase

- When you create your new graph what is changing?
  Responses may vary. Sample responses may include: The position of the curve changes; the equation of the curve changes; the intercepts change.
- When you create your new graph what is staying the same?
  Responses may vary. Sample responses may include: A linear function remains linear and a nonlinear function remains nonlinear.
- How can you determine if a relation is a function?
  Responses may vary. Determine whether there are multiple y-values for the same x-value (the vertical line test).
- Is your new graph always a function? Why or why not?
  No. For linear functions the new plot is a function; however, for the quadratic functions tested, the new plots have two different y-values for the same x-value.

Explore

The Explore portion of the lesson provides the student with an opportunity to explore concretely the concept of inverses of functions. This part of the lesson is designed for groups of three to four students.

At the end of the Explore phase, students should be able to describe the graphs of inverse relations as reflections across the line $y = x$. Students should also be able to identify whether or not two relations are inverses.

1. Distribute the **Know When to Fold ‘Em** activity sheet.

2. Prompt students to cut out each of the graphs so that they are on separate pieces of paper. Note: An alternate method is to trace the graphs onto patty paper, including the axes then fold the patty paper to find lines of symmetry.

3. Prompt students to fold the paper to find and record all possible lines of symmetry for the graphs shown and the equation for each possible line of symmetry.

4. Prompt students to select and record, in the table, the coordinates of any four points on the original graph. Then record the coordinates of the image of each point.
   *Recall that in transformations (both geometric and algebraic), the original figure is called the **pre-image** and the transformed figure is called the **image**.*

5. Facilitate the activity using the Facilitation Questions.

Facilitation Questions – Explore Phase

- How can you use paper folding to determine a line of symmetry?
  Responses may vary. Fold so that the two graphs coincide. The fold is the line of symmetry.
- What patterns do you notice in the table values?
  Responses may vary. Students should notice that the x and y values are reversed.
- How is the equation of the line of symmetry related to the pattern you discovered in your table values?
  Responses may vary. Students should connect the equation $y = x$ to the reversing of the x and y values in the tables.

Explain

The Explain portion of the lesson is directed by the teacher to allow the students to formalize their understanding of the TEKS addressed in the lesson. In this phase, debrief the ***Know When to Fold 'Em*** *activity sheet from the Explore. Use the Facilitation Questions to prompt student groups to share their responses.*

At the end of the Explain phase, students should be able to communicate multiple representations of inverses of functions and relations and use multiple methods to find and describe inverse relations.

Formalize

- The relations you explored are called inverse relations. When the domain and the range of two relations are reversed, they are called inverse relations.
- The inverse of a function may or may not be a function.
- The graph of the inverse of a function is the reflection of the function across the line $y = x$.

Facilitation Questions – Explain Phase

- How did you determine whether the relations, $y = 2x - 8$ and $y = \frac{1}{2}x + 4$ are related to each other in the same way the plots in graphs 1, 2, 3 and 4 are related to each other?
  Responses may vary. Students may have used graphs, tables, mapping or symbolic manipulation to answer the question. An example of each method is shown below.

- How did you determine whether the relations, $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$ are related to each other in the same way the plots in graphs 1, 2, 3 and 4 are related to each other?
  Responses may vary. Students may have used graphs, tables, mapping or symbolic manipulation to answer the question. An example of each method is shown below.

- When two relations are related to each other in the same way the plots in graphs 1, 2, 3 and 4 are related to each other are both relations always functions? How do you know?
  No. This is shown by counter example. For the relations, $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$, $y = \pm\sqrt{x-2}$ is not a function.

If student groups do not present all of the methods shown below, explain how they could have used the method.

Determining Inverses

Use the graphing calculator to generate the graph, table and mappings that can be used to determine if two relations are inverses.

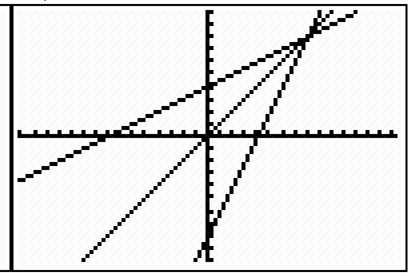
Graph

Examine the graphs of Y1 and Y2. (Hint: Be sure you are using a square viewing window.)

Note: *The graphs shown below are graphed with the line Y3 = x, which is the dotted line.*

```
WINDOW
 Xmin=-15.16129…
 Xmax=15.161290…
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```

```
 Plot1 Plot2 Plot3
\Y1=2X-8
\Y2=(1/2)X+4
\Y3=X
\Y4=
\Y5=
\Y6=
\Y7=
```

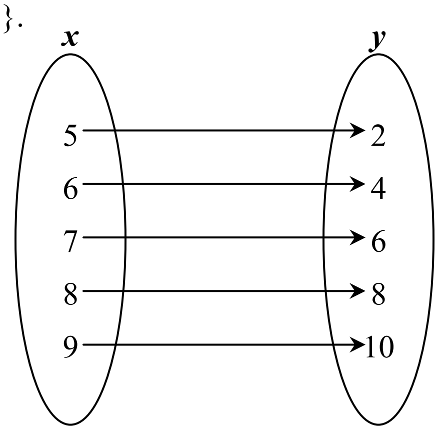


In this case the graphs of Y1 and Y2 appear to be reflections of one another across the line $y = x$ therefore they are inverses.

Table and Mapping

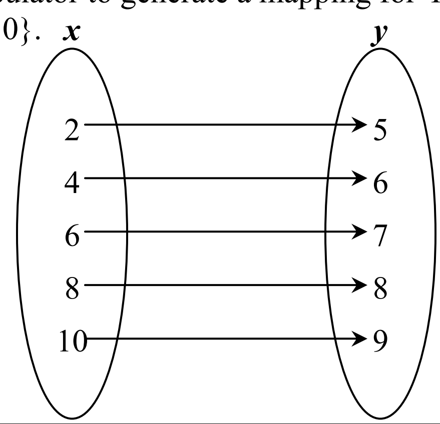
First use the table feature of the graphing calculator to generate a mapping for Y1 to show the replacement set for y when $x = \{5, 6, 7, 8, 9\}$.

X	Y1	
5	2	
6	4	
7	6	
8	8	
9	10	
10	12	
11	14	
X=11		



Next use the table feature of your graphing calculator to generate a mapping for Y2 to show the replacement set for y when $x = \{2, 4, 6, 8, 10\}$.

X	Y2	
0	4	
2	5	
4	6	
6	7	
8	8	
10	9	
12	10	
X=0		



In this case the domain and the range for the two mappings are reversed; therefore, the two relations are inverses.

Symbolically

Symbolically, x (which represents the domain values) and y (which represents the range values) are interchanged.

Original function	$y = 2x - 8$
Step 1: Interchange the variables.	$x = 2y - 8$
Step 2: Solve for y in terms of x using inverse operations.	$y = \frac{1}{2}x + 4$

Elaborate

The Elaborate portion of the lesson provides an opportunity for the student to apply the concepts of the TEKS within a new situation. This part of the lesson is designed for groups of three to four students.

At the end of the Elaborate phase, students will be able to determine the correct representations of function inverses and whether the inverse of a function is also a function.

1. Distribute the **Multiple Representations of Inverses** activity sheet. Students should follow the directions to solve the problems and answer the questions.

2. Use the Facilitation Questions to redirect students as necessary**.**

Facilitation Questions – Elaborate Phase

- How can you determine the inverse of a function?
  Students could reflect the graph of the function across the line y = x then determine the equation of the reflected line.
  Symbolically, students could reverse the domain (x) and range (y) then solve for y using inverse operations.

- How can you determine if the inverse of a function is a function?
  Responses may vary. Determine whether there are multiple y-values for the same x-value (the vertical line test).

- Given two functions, how can you determine if one is the inverse of the other?
  Responses may vary. Students could graph each relation and determine if y = x is a line of symmetry, use the table feature on the calculator to compare mappings, or reverse the domain (x) and range (y) then solve for y using inverse operations.

Evaluate

The Evaluate portion of the lesson provides the student with an opportunity to demonstrate his or her understanding of the TEKS addressed in the lesson.

1. Provide each student with the **How Cheesy** performance assessment.

2. Provide a time limit for completion of the assessment. Students should justify their answers through graphs, tables, and/or algebraic methods.

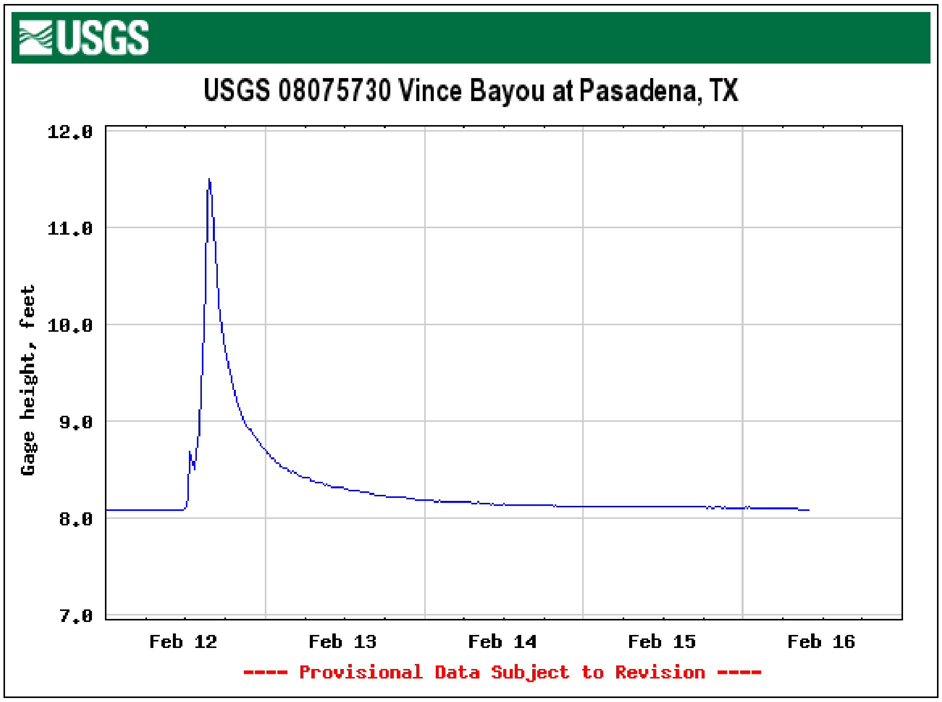
3. Upon completion of the assessment, a rubric should be used to assess student understanding of the concepts addressed in the lesson.

Rising Water

Answer Key

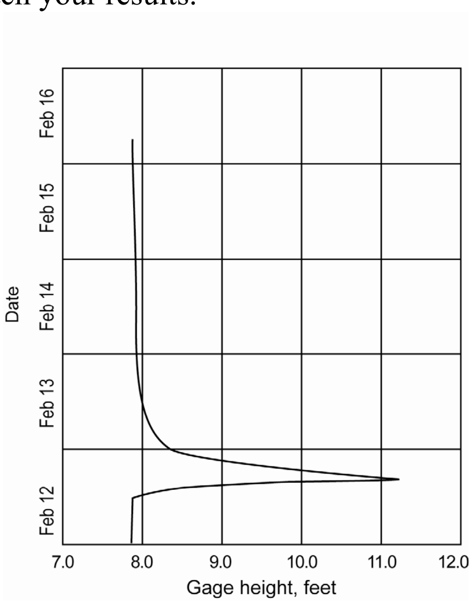
The United States Geological Survey (USGS) maintains stream flow data for rivers, creeks, and streams across the United States. One type of data they record is the gage height of a particular stream. Gage height is the height on a gage above a given zero-point.

The following graph shows gage height over time for Vince Bayou near Pasadena, Texas, for February 2007.



If the date were on the vertical axis and the gage height were on the horizontal axis, what would the new graph look like? Sketch your results.

Possible Response:



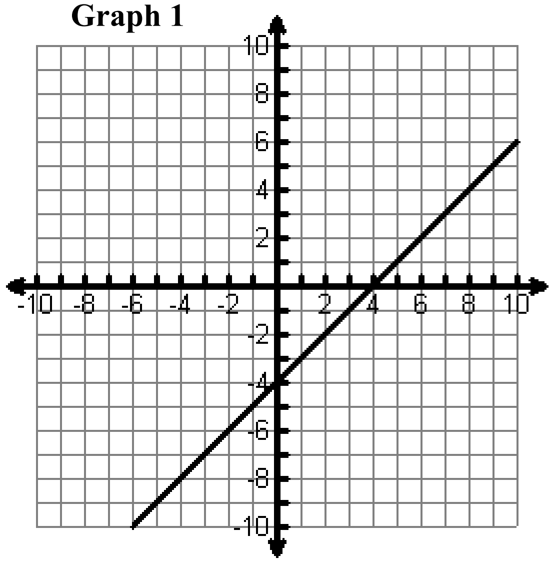
Know When to Hold 'Em

Answer Key

For each graph, trace the graph onto a sheet of patty paper. Be sure to trace the plot and the axes then label your axes on your patty paper.

Take the top-right corner of your patty paper in your right hand and the bottom-left corner in your left hand and flip the patty paper over so that the top-left corner and the bottom-right corner of the patty paper change places. Align the origin on the patty paper with the origin of your original graph and align the axes on the patty paper with the axes on the graph. While observing the original position and the new position of the plot, answer the related questions.

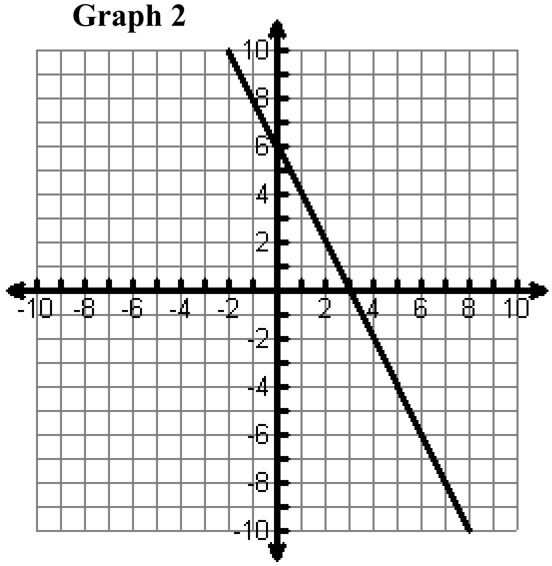
Graph 1



1. How is the new graph similar to the original graph? How are they different?
   Responses may vary. Sample responses may include: The position of the graph changes; the equation of the graph changes; the intercepts change. The domain became the range; the range became the domain.

2. Is the new graph a function? How do you know?
   Yes. The new graph is a function since for each y-value there is only one x-value (the new graph passes the vertical line test).

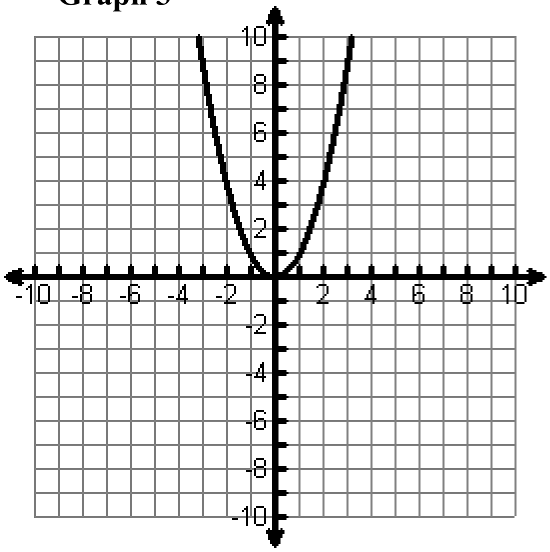
Graph 2



3. How is the new graph similar to the original graph? How are they different?
   Responses may vary. Sample responses may include: The position of the graph changes; the equation of the graph changes; the intercepts change. The domain became the range; the range became the domain.

4. Is the new graph a function? How do you know?
   Yes. The new graph is a function since for each y-value there is only one x-value (the new graph passes the vertical line test).

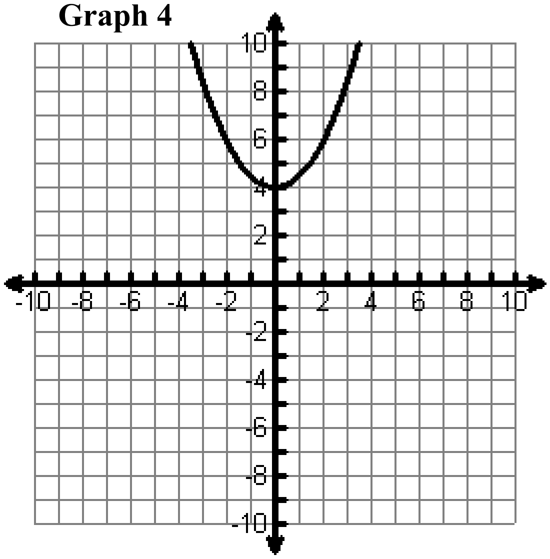
Graph 3



5. How is the new graph similar to the original graph? How are they different?
   Responses may vary. Sample responses may include: The position of the graph changes; the equation of the graph changes; the domain became the range; the range became the domain.

6. Is the new graph a function? How do you know?
   The new graph is not a function because there are multiple y-values for the same x-value (it fails the vertical line test).

Graph 4

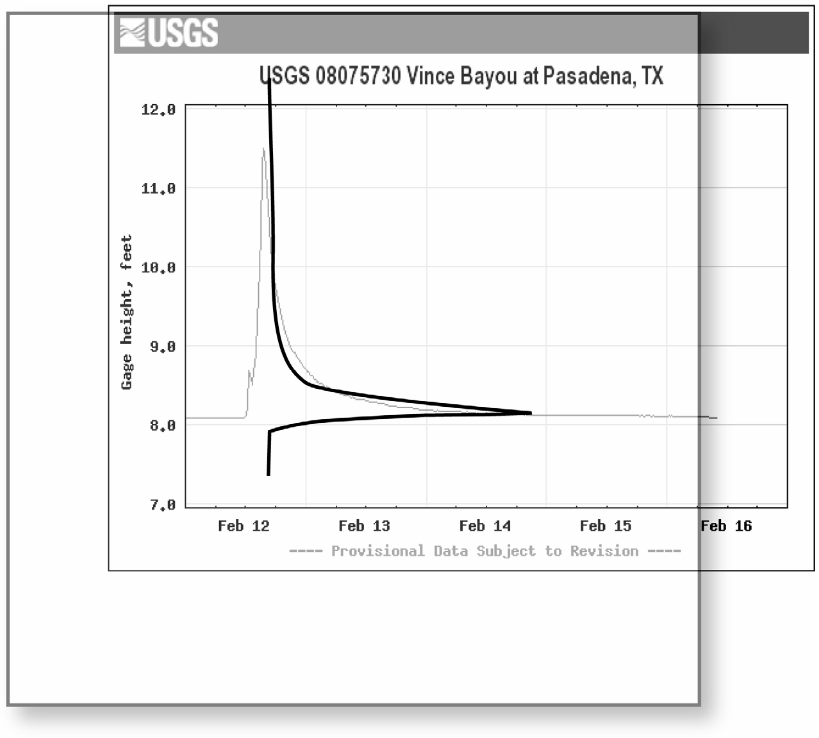


7. How is the new graph similar to the original graph? How are they different?
   Responses may vary. Sample responses may include: The position of the graph changes; the equation of the graph changes; the domain became the range; the range became the domain.

8. Is the new graph a function? How do you know?
   The new graph is not a function because there are multiple y-values for the same x-value (it fails the vertical line test).

9. What characteristics do all four new graphs have in common?
   Responses may vary. Sample responses may include: For all graphs the position of the original graph changes; the equation of the original graph changes; the domain became the range; the range became the domain.

10. Revisit the graph from **Rising Water**. Trace the graph and both axes (don't forget to label your axes) onto a sheet of patty paper then flip the traced graph just like you did with Graphs 1 – 4. Sketch your results.
Possible Response:



11. How does your flipped graph compare to your prediction from **Rising Water**? Explain any similarities or differences.
Responses may vary.

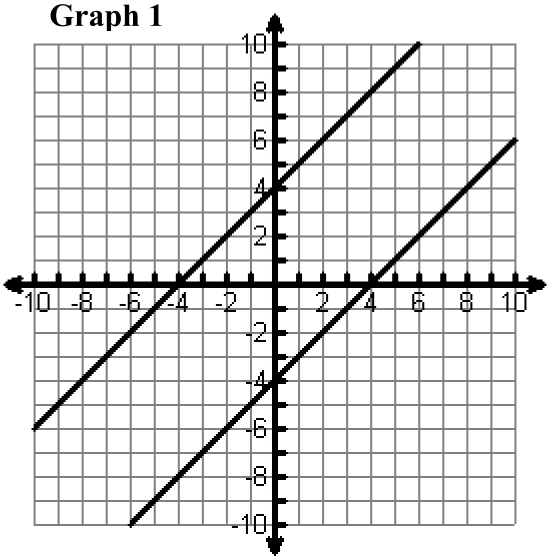
Know When to Fold 'Em

Answer Key

Cut out each of the graphs so that they are on separate pieces of paper. For each graph, fold the paper to find all possible lines of symmetry for the graphs shown. Identify the equation for each possible line of symmetry. In the table record the coordinates of four points on the original graph. Then record the coordinates of the image of each point.

TIP: It may be easier to trace the graphs onto patty paper, including the axes. Fold the patty paper to find lines of symmetry.

Graph 1

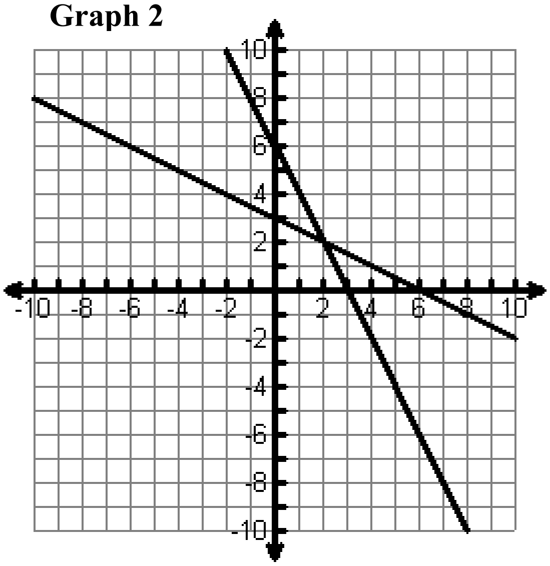


1. What is/are the equation(s) of the line(s) of symmetry for Graph 1?
   Responses may vary. Students should identify the line $y = x$.

2. Complete the table.
   Sample values are given.

Original			Image	
x	***y***		***x***	***y***
−5	*−9*		*−9*	*−5*
−3	*−7*		*−7*	*−3*
−1	*−5*		*−5*	*−1*
1	*−3*		*−3*	*1*

Graph 2

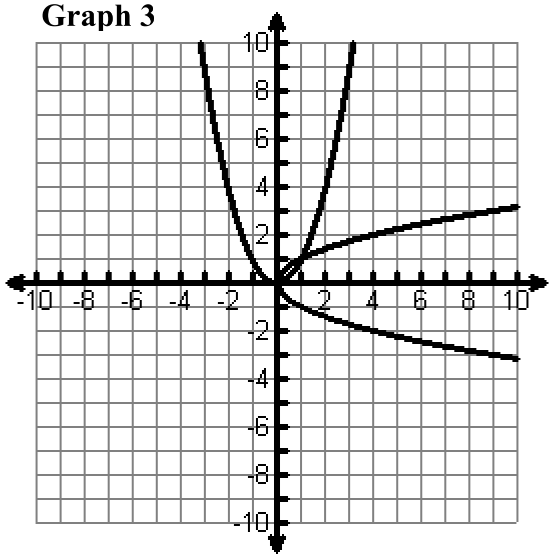


3. What is/are the equation(s) of the line(s) of symmetry for Graph 2?
   Responses may vary. Students should identify the line $y = x$.

4. Complete the table.
   Sample values are given.

Original			Image	
x	***y***		***x***	***y***
−2	*10*		*10*	*−2*
2	*2*		*2*	*2*
4	*−2*		*−2*	*4*
8	*−10*		*−10*	*8*

Graph 3

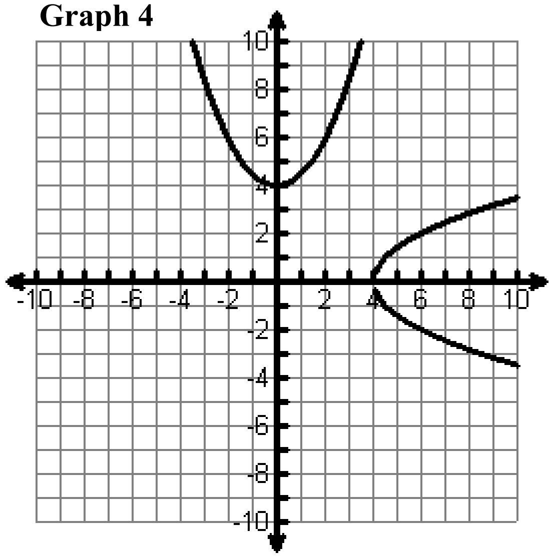


5. What is/are the equation(s) of the line(s) of symmetry for Graph 3?
Responses may vary. Students should identify the line $y = x$.

6. Complete the table.
Sample values are given.

Original			Image	
x	**y**		**x**	**y**
−3	9		9	−3
−1	1		1	−1
1	1		1	1
3	9		9	3

Graph 4



7. What is/are the equation(s) of the line(s) of symmetry for Graph 4?
Responses may vary. Students should identify the line $y = x$.

8. Complete the table.
Sample values are given.

Original			Image	
x	**y**		**x**	**y**
−2	6		6	−2
−1	4.5		4.5	−1
1	4.5		4.5	1
2	6		6	2

9. What patterns do you notice among the lines of symmetry for each of the graphs?
Responses may vary. Students should notice that for each of the given pairs of graphs, the line $y = x$ is a line of symmetry.

10. Which transformation describes the folds across a line of symmetry for your graphs?
Each fold across a line of symmetry reveals a reflection across the line of symmetry.

11. How are the *x*- and *y*-coordinates from the original related to the *x*- and *y*-coordinates of its image? How could you represent this relationship symbolically?
The x-coordinates from the first graph become the y-coordinates for the second graph, and the y-coordinates from the first graph become the x-coordinates for the second graph.

Symbolically, this relationship can be represented using the transformation mapping $T:(x,\ y) \to (y,\ x)$.

12. Are the relations, $y = 2x - 8$ and $y = \frac{1}{2}x + 4$ related to each other in the same way as graphs 1, 2, 3 and 4 are related to each other? How do you know?
Yes. Students may have graphed each relation and determine if $y = x$ is a line of symmetry or used the table feature on the calculator to compare mappings.

13. Are the relations, $y = 2x - 8$ and $y = \frac{1}{2}x + 4$ both functions? How do you know?
Yes. Both relations are functions since in both cases for each y-value, there is only one x-value.

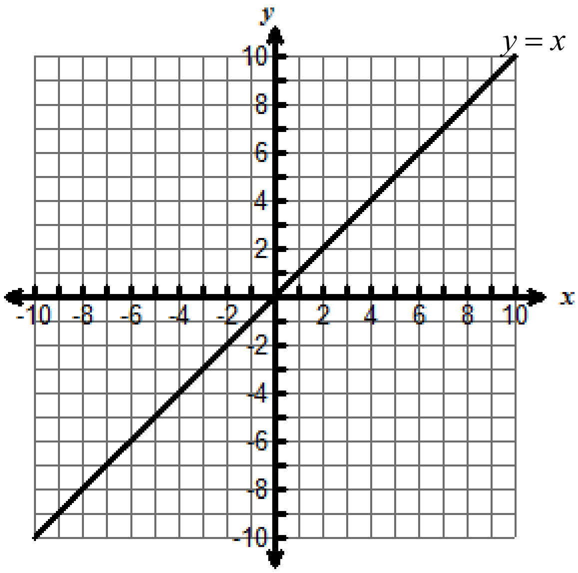
14. Are the relations, $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$ related to each other in the same way the plots in graphs 1, 2, 3 and 4 are related to each other? How do you know?
Yes. Students may have graphed each relation and determine if $y = x$ is a line of symmetry or used the table feature on the calculator to compare mappings.

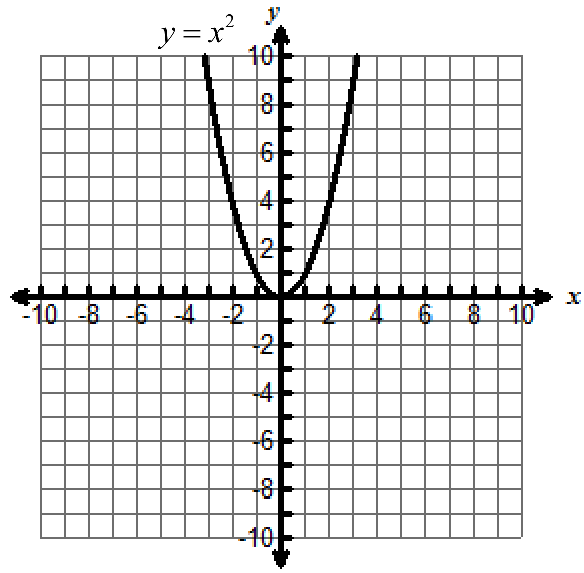
15. Are the relations, $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$ both functions? How do you know?
The relation $y = x^2 + 2$ is a function; however, the relation $y = \pm\sqrt{x-2}$ is not a function because there are multiple y-values for the same x-value (i.e., it fails the vertical line test).

Multiple Representations of Inverses

Answer Key

The graphs of two parent functions are shown.

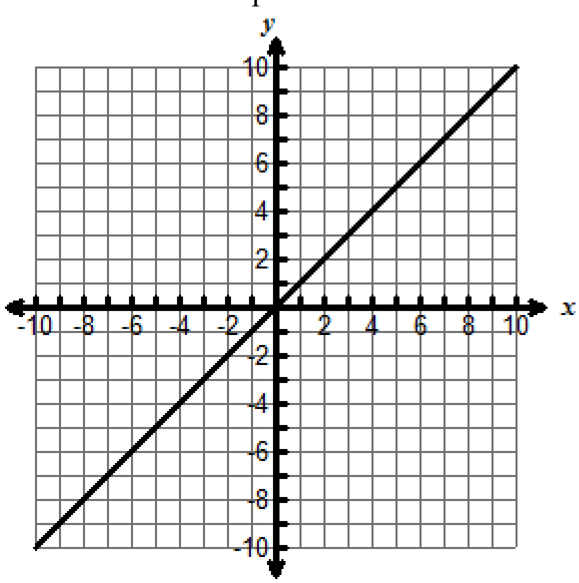




Trace each parent function onto a separate piece of patty paper. Be sure to trace and label the axes as well.

Linear Functions:

1. Reflect the linear parent function across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of the inverse of the linear parent function? How do they compare with the original function?
   The domain and range are both all real numbers, just like the original parent function.

3. Is the inverse of the linear parent function also a function? How do you know?
   Yes, the inverse is also a function since for each x-value there is only one corresponding y-value.

4. What kind of function is the inverse of a linear function?
   The inverse is also a linear function.

5. What is the inverse of the function $y = \frac{2}{5}x - 7$?

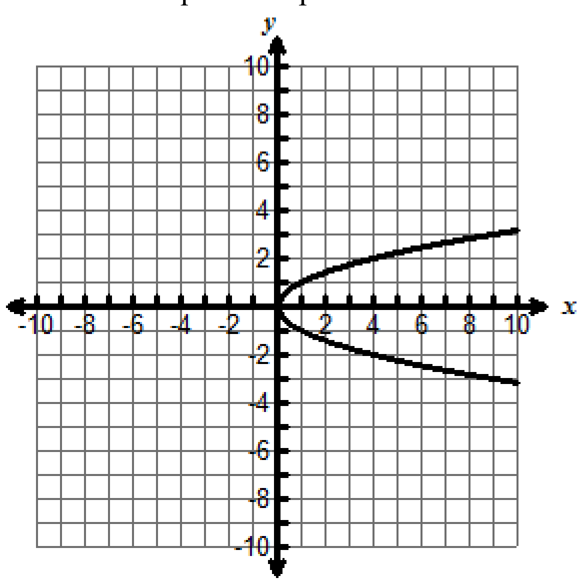
   $y = \frac{5}{2}(x+7)$ *or* $y = \frac{5}{2}x + \frac{35}{2}$

6. How did you determine the inverse?
   Students could reflect the graph of the function across the line y = x then determine the equation of the reflected line.

   Symbolically, students may reverse the domain (x) and range (y) then solve for y using inverse operations.

Quadratic Functions

7. Reflect the quadratic parent function across the line $y = x$. Sketch your resulting graph.



8. What is the domain and range of the inverse of the quadratic parent function? How do they compare with the original function?
   The domain is {x: x ≥ 0}; the range is all real numbers.
   The domain of the original parent function became the range of the inverse.
   The range of the original parent function became the domain of the inverse.

9. Is the inverse of the quadratic parent function also a function? How do you know?
   No. For most x-values (all except x = 0), there are two corresponding y-values. For example, when x = 4, y = +2 and −2.

10. If the inverse is not a function, how can we restrict the domain and/or range so that the inverse is a function?
    If we only consider the range {y: y ≥ 0}, then the inverse becomes a function.

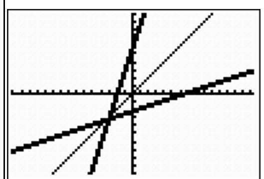
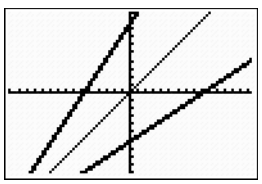
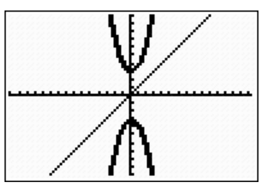
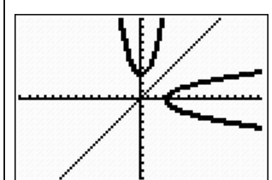
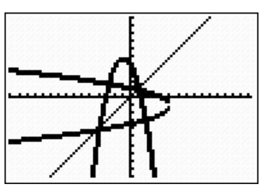
11. What kind of function is the inverse of a quadratic function?
    The inverse of a quadratic function is a square root function.

12. What is the inverse function of the function $y = -3(x-1)^2 + 2$?
    $y = 1 + \sqrt{\dfrac{x-2}{-3}}$ *or* $y = 1 - \sqrt{\dfrac{x-2}{-3}}$
    Typically, the principal (positive) square root is used as the inverse of a quadratic function.

13. How did you determine the inverse?
    Students could reflect the graph of the function across the line y = x then determine the equation of the reflected curve.

    Symbolically, students may reverse the domain (x) and range (y) then solve for y using inverse operations.

14. For each function and its inverse, one of the given representations is incorrect. Cross out the incorrect representation then create it correctly in the last column.

Symbolic	**Graph**	**Mappings**		**Corrected Representation**
$y = 3x + 6$ and $y = \frac{1}{3}x - 6$		$x \to y$ $2 \to 12$ $4 \to 18$ $6 \to 24$ $8 \to 30$	$x \to y$ $12 \to 2$ $18 \to 4$ $24 \to 6$ $30 \to 8$	$y = 3x + 6$ and $y = \frac{1}{3}x - 2$
$y = \frac{2}{3}x - 6$ and $y = \frac{3}{2}x + 9$		$x \to y$ $0 \to -8$ $3 \to -6$ $6 \to -4$ $9 \to -2$	$x \to y$ $-8 \to 0$ $-6 \to 3$ $-4 \to 6$ $-2 \to 9$	$x \to y$ $0 \to -6$ $3 \to -4$ $6 \to -2$ $9 \to 0$ $x \to y$ $-6 \to 0$ $-4 \to 3$ $-2 \to 6$ $0 \to 9$
$y = x^2 + 3$ and $y = \pm\sqrt{x - 3}$		$x \to y$ $2 \to 7$ $4 \to 19$ $6 \to 39$ $8 \to 67$	$x \to y$ $7 \to 2$ $19 \to 4$ $39 \to 6$ $67 \to 8$	
$y = -x^2 - 2x + 4$ and $y = -1 \pm \sqrt{-x + 3}$		$x \to y$ $2 \to -4$ $4 \to -20$ $6 \to -44$ $8 \to -76$	$x \to y$ $-4 \to 2$ $-20 \to 4$ $-44 \to 6$ $-76 \to 8$	$y = -x^2 - 2x + 4$ and $y = -1 \pm \sqrt{-x + 5}$

How Cheesy

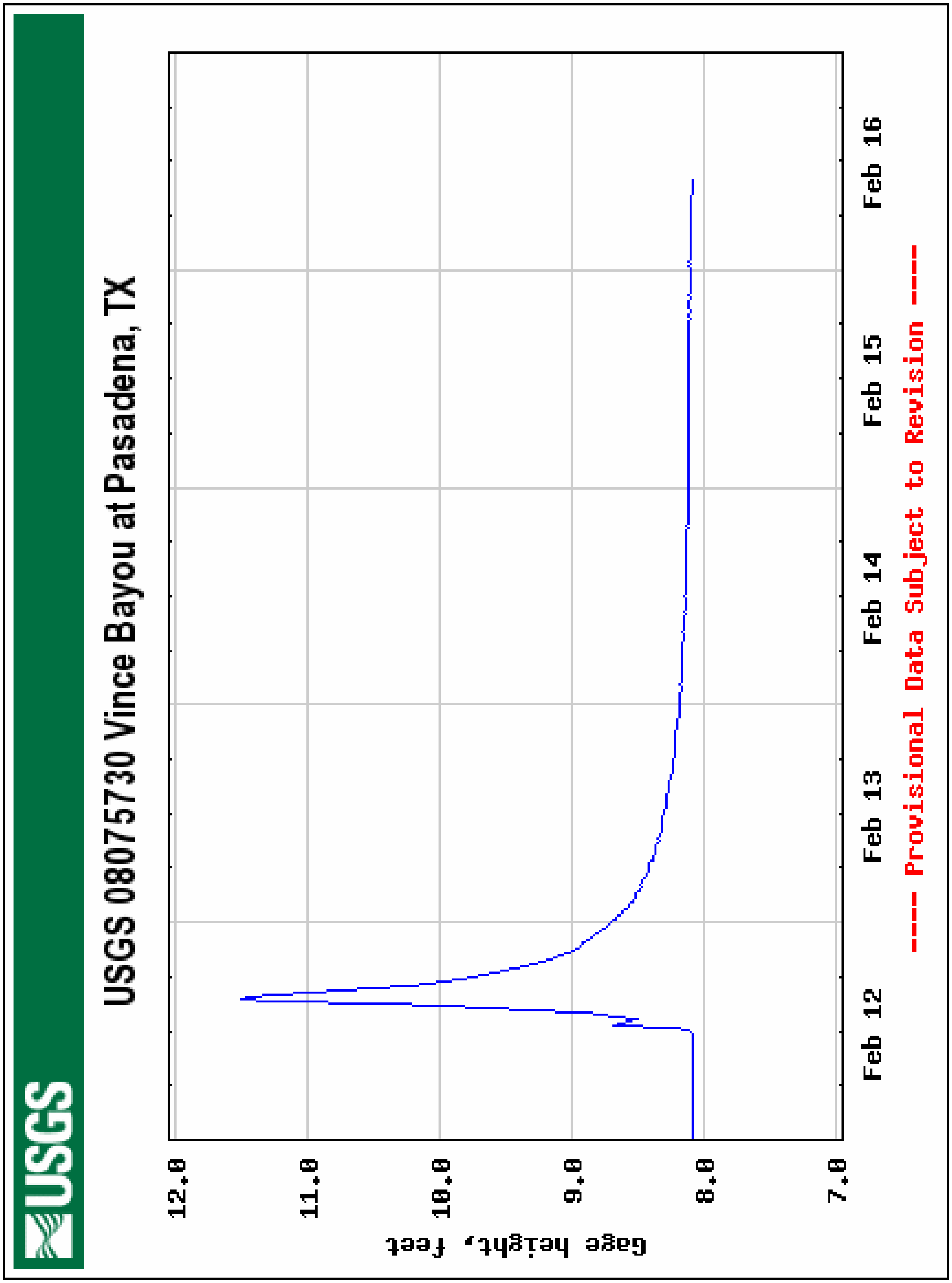
Answer Key

At the Cheesy Cheese Cake Store the charge to deliver a cheese cake to a party is determined by the number of miles the party is from the Cheesy Cheese Cake Store. There is fixed administrative charge of \$10 plus \$0.45 for each mile driven one way. The amount charged can be modeled by the function $y = 0.45x + 10$ where x is the number of miles driven one way and y is the total charge. On a recent delivery Craig charged a customer \$19.90, but when he went to fill out his delivery log, he could not remember the number of miles he drove.

How many miles did Craig drive? Justify your answer using the inverse of the function $y = 0.45x + 10$.

Craig drove 22 miles.

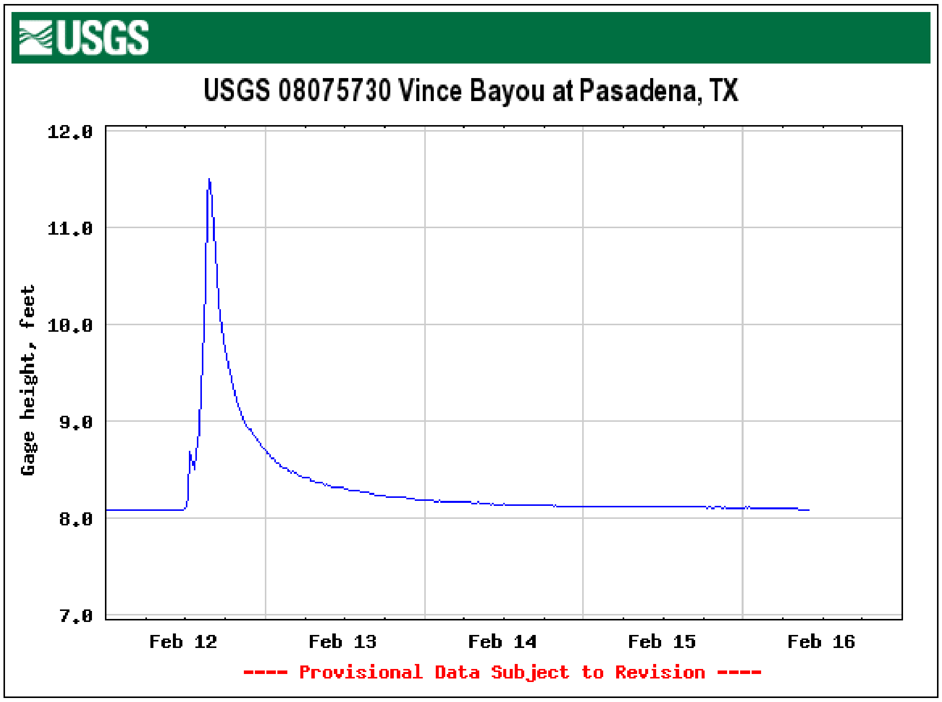
Transparency: Vince Bayou



Rising Water

The United States Geological Survey (USGS) maintains stream flow data for rivers, creeks, and streams across the United States. One type of data they record is the gage height of a particular stream. Gage height is the height on a gage above a given zero-point.

The following graph shows gage height over time for Vince Bayou near Pasadena, Texas, for February 2007.



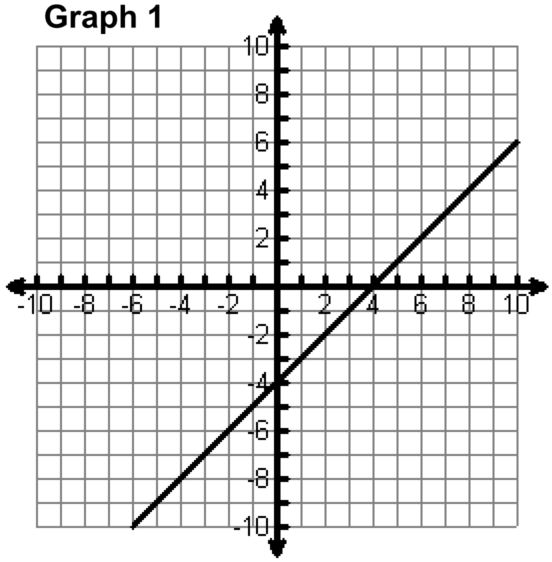
If the date were on the vertical axis and the gage height were on the horizontal axis, what would the new graph look like? Sketch your results.

Know When to Hold 'Em

For each graph, trace the graph onto a sheet of patty paper. Be sure to trace the plot and the axes then label your axes on your patty paper.

Take the top-right corner of your patty paper in your right hand and the bottom-left corner in your left hand and flip the patty paper over so that the top-left corner and the bottom-right corner of the patty paper change places. Align the origin on the patty paper with the origin of your original graph and align the axes on the patty paper with the axes on the graph. While observing the original position and the new position of the plot, answer the related questions.

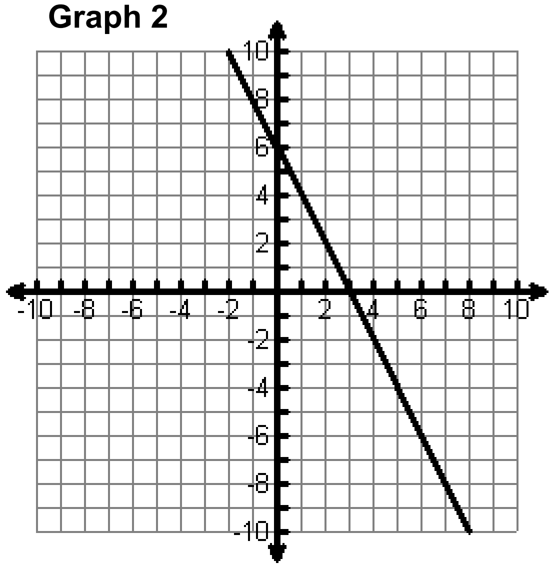
Graph 1



1. How is the new graph similar to the original graph? How are they different?

2. Is the new graph a function? How do you know?

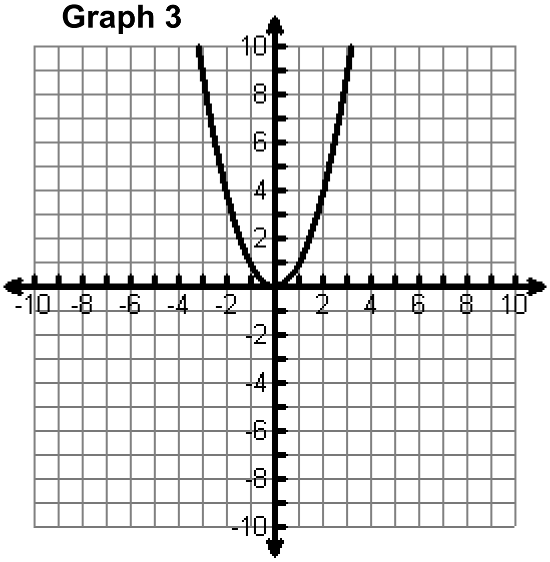
Graph 2



3. How is the new graph similar to the original graph? How are they different?

4. Is the new graph a function? How do you know?

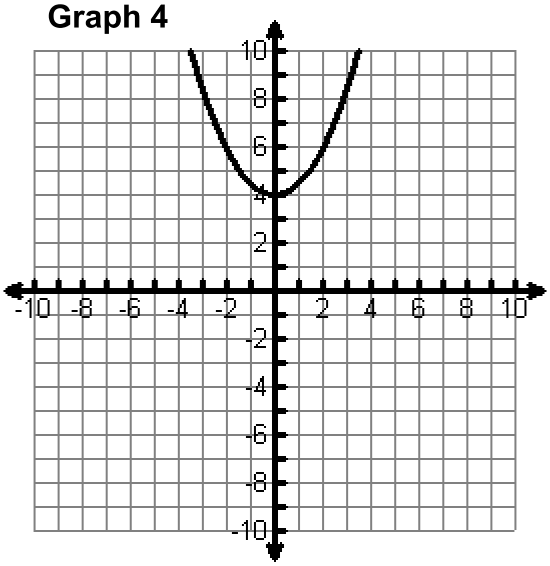
Graph 3



5. How is the new graph similar to the original graph? How are they different?

6. Is the new graph a function? How do you know?

Graph 4



7. How is the new graph similar to the original graph? How are they different?

8. Is the new graph a function? How do you know?

9. What characteristics do all four new graphs have in common?

10. Revisit the graph from **Rising Water**. Trace the graph and both axes (don't forget to label your axes) onto a sheet of patty paper then flip the traced graph just like you did with Graphs 1 – 4. Sketch your results.

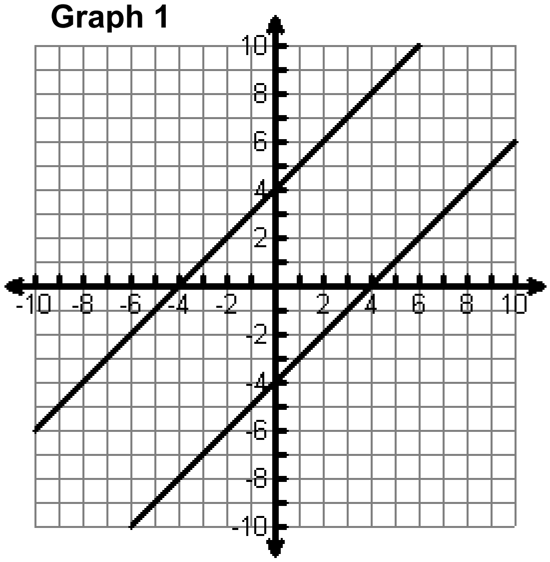
11. How does your flipped graph compare to your prediction from **Rising Water**? Explain any similarities or differences.

Know When to Fold 'Em

Cut out each of the graphs so that they are on separate pieces of paper. For each graph, fold the paper to find all possible lines of symmetry for the graphs shown. Identify the equation for each possible line of symmetry. In the table record the coordinates of four points on the original graph. Then record the coordinates of the image of each point.

TIP: It may be easier to trace the graphs onto patty paper, including the axes. Fold the patty paper to find lines of symmetry.

Graph 1

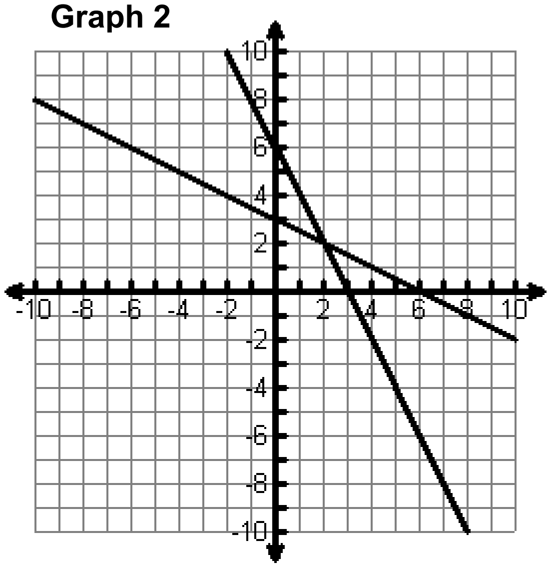


1. What is/are the equation(s) of the line(s) of symmetry for Graph 1?

2. Complete the table.

Original			Image	
x	*y*		*x*	*y*

Graph 2

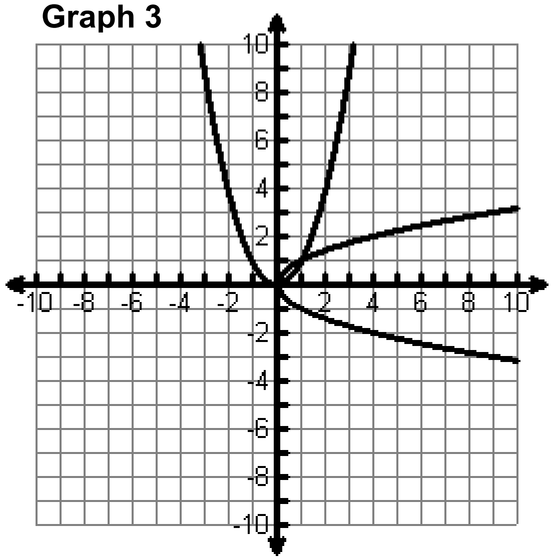


3. What is/are the equation(s) of the line(s) of symmetry for Graph 2?

4. Complete the table.

Original			Image	
x	*y*		*x*	*y*

Graph 3

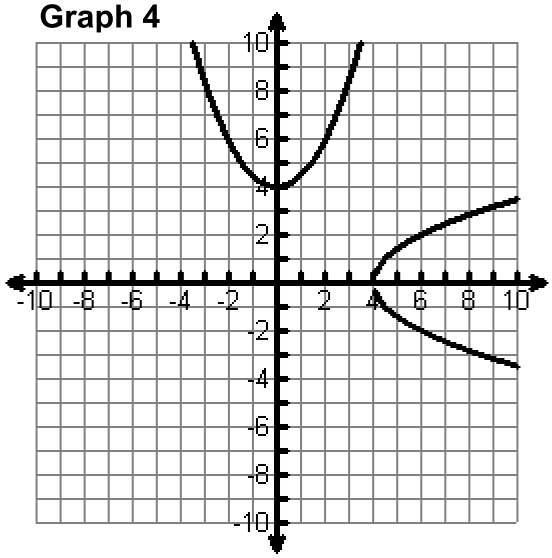


5. What is/are the equation(s) of the line(s) of symmetry for Graph 3?

6. Complete the table.

Original			Image	
x	***y***		***x***	***y***

Graph 4



7. What is/are the equation(s) of the line(s) of symmetry for Graph 4?

8. Complete the table.

Original			Image	
x	***y***		***x***	***y***

9. What patterns do you notice among the lines of symmetry for each of the graphs?

10. Which transformation describes the folds across a line of symmetry for your graphs?

11. How are the x- and y-coordinates from the original related to the x- and y-coordinates of its image? How could you represent this relationship symbolically?

12. Are the relations, $y = 2x - 8$ and $y = \frac{1}{2}x + 4$ related to each other in the same way the plots in graphs 1, 2, 3 and 4 are related to each other? How do you know?

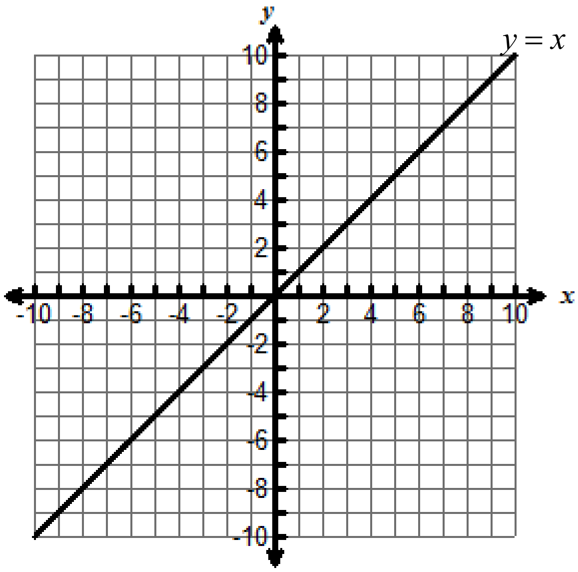
13. Are the relations, $y = 2x - 8$ and $y = \frac{1}{2}x + 4$ both functions? How do you know?

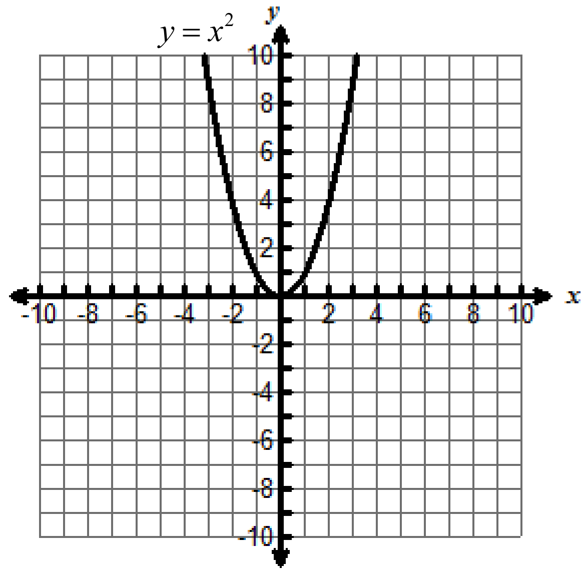
14. Are the relations, $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$ related to each other in the same way the plots in graphs 1, 2, 3 and 4 are related to each other? How do you know?

15. Are the relations, $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$ both functions? How do you know?

Multiple Representations of Inverses

The graphs of two parent functions are shown.

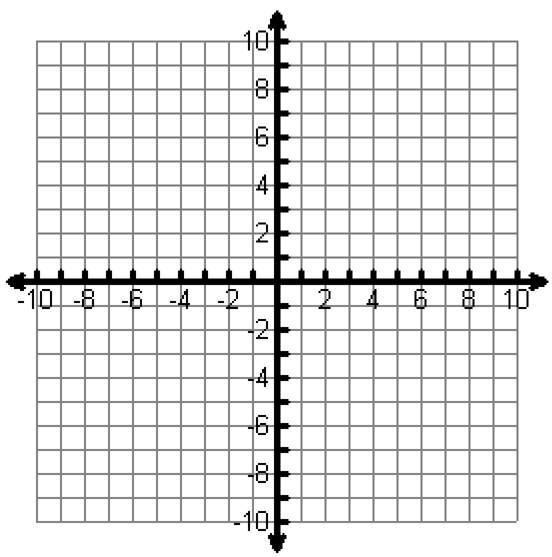




Trace each parent function onto a separate piece of patty paper. Be sure to trace and label the axes as well.

Linear Functions:

1. Reflect the linear parent function across the line $y = x$. Sketch your resulting graph.



2. What is the domain and range of the inverse of the linear parent function? How do they compare with the original function?

3. Is the inverse of the linear parent function also a function? How do you know?

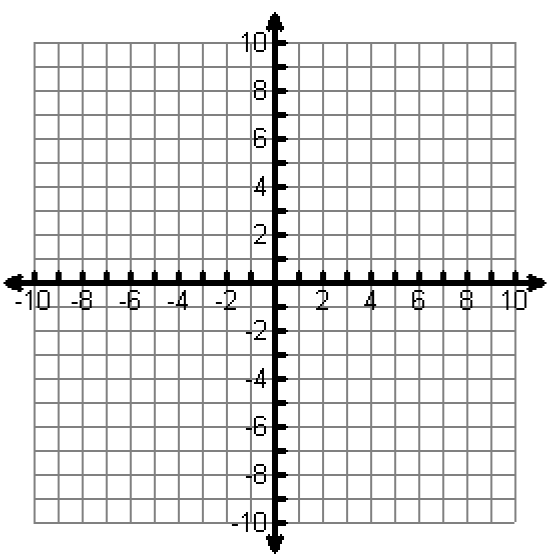
4. What kind of function is the inverse of a linear function?

5. What is the inverse of the function $y = \frac{2}{5}x - 7$?

6. How did you determine the inverse?

Quadratic Functions

7. Reflect the quadratic parent function across the line $y = x$. Sketch your resulting graph.



8. What is the domain and range of the inverse of the quadratic parent function? How do they compare with the original function?

9. Is the inverse of the quadratic parent function also a function? How do you know?

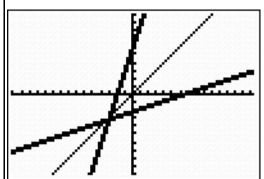
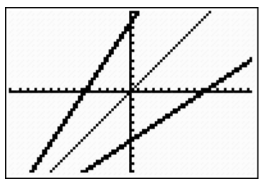
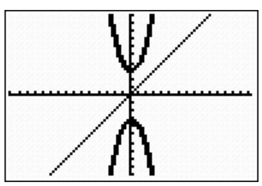
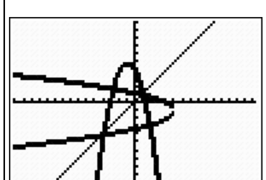
10. If the inverse is not a function, how can we restrict the domain and/or range so that the inverse is a function?

11. What kind of function is the inverse of a quadratic function?

12. What is the inverse of the function $y = -3(x-1)^2 + 2$?

13. How did you determine the inverse?

14. For each function and its inverse, one of the given representations is incorrect. Cross out the incorrect representation then create it correctly in the last column.

Symbolic	Graph	Mappings		Corrected Representation
$y = 3x + 6$ and $y = \frac{1}{3}x - 6$		$x \to y$: $2 \to 12$, $4 \to 18$, $6 \to 24$, $8 \to 30$	$x \to y$: $12 \to 2$, $18 \to 4$, $24 \to 6$, $30 \to 8$	
$y = \frac{2}{3}x - 6$ and $y = \frac{3}{2}x + 9$		$x \to y$: $0 \to -8$, $3 \to -6$, $6 \to -4$, $9 \to -2$	$x \to y$: $-8 \to 0$, $-6 \to 3$, $-4 \to 6$, $-2 \to 9$	
$y = x^2 + 3$ and $y = \pm\sqrt{x - 3}$		$x \to y$: $2 \to 7$, $4 \to 19$, $6 \to 39$, $8 \to 67$	$x \to y$: $7 \to 2$, $19 \to 4$, $39 \to 6$, $67 \to 8$	
$y = -x^2 - 2x + 4$ and $y = -1 \pm \sqrt{-x + 3}$		$x \to y$: $2 \to -4$, $4 \to -20$, $6 \to -44$, $8 \to -76$	$x \to y$: $-4 \to 2$, $-20 \to 4$, $-44 \to 6$, $-76 \to 8$	

How Cheesy

At the Cheesy Cheese Cake Store the charge to deliver a cheese cake to a party is determined by the number of miles the party is from the Cheesy Cheese Cake Store. There is fixed administrative charge of $10 plus $0.45 for each mile driven one way. The amount charged can be modeled by the function $y = 0.45x + 10$ where x is the number of miles driven one way and y is the total charge. On a recent delivery Craig charged a customer $19.90, but when he went to fill out his delivery log, he could not remember the number of miles he drove.

How many miles did Craig drive? Justify your answer using the inverse of the function $y = 0.45x + 10$.

Student Lesson: Square Root Functions

TEKS

a5 Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.9 **Quadratic and Square Root Functions**. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

2A.9A The student is expected to use the parent function to investigate, describe, and predict the effects of parameter changes on the graphs of square root functions and describe limitations on the domains and ranges.

2A.9B The student is expected to relate representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

2A.9C The student is expected to determine the reasonable domain and range values of square root functions, as well as interpret and determine the reasonableness of solutions to square root equations and inequalities.

2A.9D The student is expected to determine solutions of square root equations using graphs, tables, and algebraic methods.

2A.9E The student is expected to determine solutions of square root inequalities using graphs and tables.

2A.9F The student is expected to analyze situations modeled by square root functions, formulate equations or inequalities, select a method, and solve problems.

2A.9G The student is expected to connect inverses of square root functions with quadratic functions.

Materials

Prepare in Advance:

- Make copies of student pages and overhead transparencies.
- Collect materials for **Bottle Bounce** activity.
- Cut cards for the **Card Sort** in Activity 3 and place each set in a bag or envelope.

Presenter Materials:

- Metronome for demonstration, OR equipment to show the video clip of the metronome

For each student group of 3 - 4 students:

- Plastic 16-ounce bottle, filled with water
- Rubber bands (enough to loop together to make a "spring" about 1 meter long)
- Metric tape measure or meter stick
- Stopwatch
- Chart paper and markers, or transparencies and overhead pens
- 1 **Card Sort** set per group (Activity 3)
- 1 copy per group of **The Pirate Ship**

For each student:

- Graphing calculator
- **Bottle Bounce** Activity 1
- **Represent Square Root Functions** Activity 3
- **The Pirate Ship**
- **Selected Response Items**

Engage

The Engage portion of the lesson is designed to generate student interest in the square root relationship found in the Bottle Bounce Activity. This part of the lesson is designed for groups of 4 students.

1. Display **Transparency: Metronome**. Ask a student with band or choir experience to explain to the class what a metronome is.
2. Demonstrate the use of a metronome. First, show students how a metronome keeps time for a slow tempo (largo). Then, show students how a metronome keeps time for a fast tempo (allegro).
3. Ask students to compare how the metronome keeps time for the two tempos. Record responses on the Transparency.
   Possible responses may include:
   For the slow tempo, the metronome moved more slowly than for the fast tempo.
   For the slow tempo, the weight is further toward the end of the metronome than for the fast tempo.

Explore

The Explore portion of the lesson provides the student with an opportunity to be actively involved in the exploration of the mathematical concepts addressed. This part of the lesson will be a combination of whole group and small group format.

1. Distribute the **Bottle Bounce Activity 1** activity sheet, and display **Transparency #1**.
2. Discuss the situation, steps, and group jobs. Check for understanding and clarify as needed.
3. Students will work together to collect data and answer the questions in **Part A** of the activity.
4. Instruct students to display their table and graph of their data on chart paper (or on group copies of **Transparencies #2 and #3**).
5. Debrief **Part A** of **Bottle Bounce Activity 1** activity sheet by discussing the similarities and differences in the tables of data and graphs created by the various groups.
6. Circulate about the room, asking groups questions in order to assess or guide. Periodically call students back as a whole class to discuss group responses.

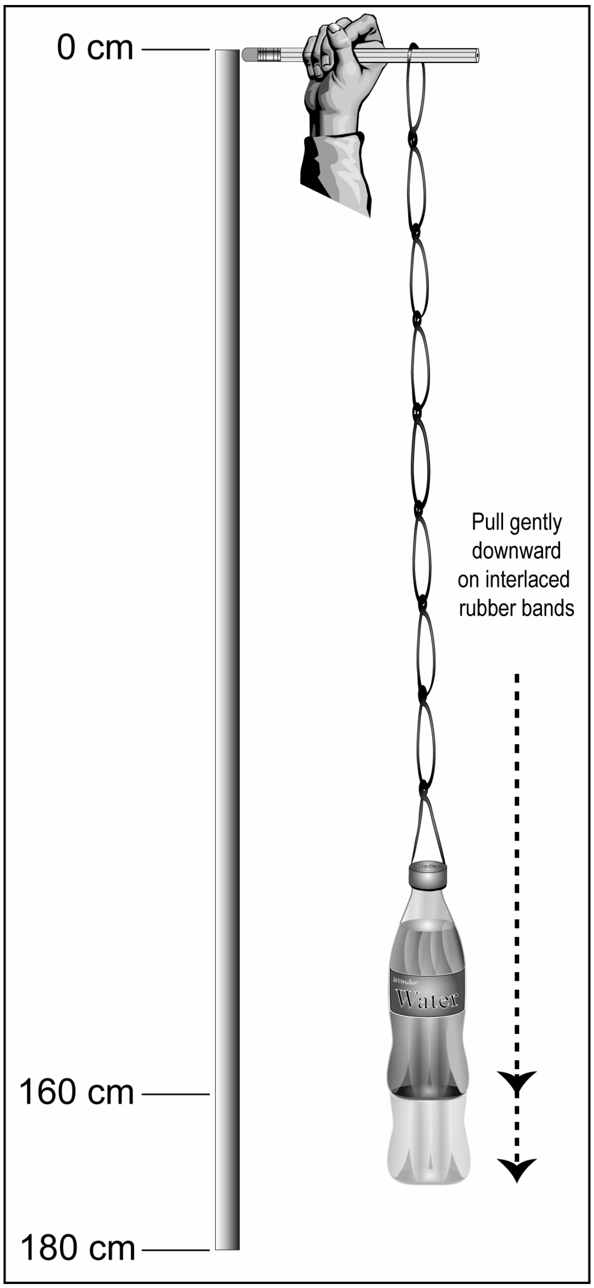
Facilitation Questions – Explore, Part A

- What is happening to the time interval as the length of the spring increases?
  The time also increases, though not at a constant rate.
- What factor(s) limit the length of the rubber band "spring"?
  We are limited by how high we can hold the pencil to support the spring.
- What time did you get when the bottom of the bottle was 0 cm from the pencil "spring" length of 0?
  It could not be done because there was no rubber band left to allow movement.
- Are the data points from all groups' graphs increasing? Explain.
  They should be. As the length of the spring increases, the interval time increased.
- How is this graph different from graphs of other functions?
  Answers will vary. The shape seems like half of a parabola on its side.
- How is this graph similar to graphs of other functions?
  Answers will vary. Just as with some other functions, it is increasing.
- Did all groups get the exact same data? Explain.
  No. There are other variables (besides just time and length of spring) that affect the outcome.

Bottle Bounce
Activity 1
Answer Key

The data collection activity that follows will model the movement of the metronome by letting a filled bottle bounce up and down.

A filled 16-ounce bottle will be suspended from a pencil by a 1-meter strand of rubber bands. When the bottle is attached the stretched rubber band spring should extend so that the bottom of the bottle of water is at least 140 cm below the pencil.



One person will hold the pencil (eraser end held firmly against the wall) at a height of 160 to 200 centimeters above the floor. The rubber band spring should be attached close to the opposite end of the pencil so the bottle can bounce without striking the wall. To simulate moving the weight up or down on the metronome, the rubber band spring will be wrapped around the pencil until the bottom of the bottle is raised to the desired height.

Collecting Data
In this investigation, a metric tape measure should be taped to the wall. You will suspend a filled 16-ounce bottle by a 1-meter rubber band spring from a pencil firmly held perpendicular to a wall at a height of 160-200 meters. The difference between the length of the rubber band spring and the suspension height should allow the bottle to bounce without touching the floor.

The length of the rubber band spring will be shortened by wrapping the spring around the pencil so that data is collected at 20-centimeter intervals.

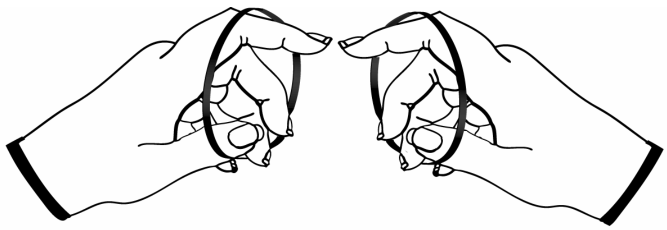
You will be working in groups of 4. Each person in the group has a job.

Materials manager:	Gets the necessary materials, directs the team in setting up the investigation, holds the pencil with the suspended bottle, and shortens the spring when needed.
Measure manager:	Measures the distance from the pencil to the bottom of the bottle for each length, initiates the bounce by pulling the suspended bottle down an additional 10 centimeters and counts the bounces (10 at each height).
Time manager:	Uses a stop watch to determine the length of each 10-bounce period of time. The time starts when the bottle is released by the measures manager and ends when the bottle completes its 10^{th} bounce.
Data manager:	Records the necessary measurements in the table and shares the data with the team.

Set-up Instructions

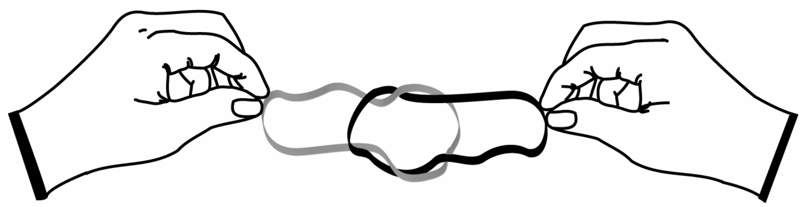
Step 1 The materials manager should get the necessary materials and ask two of the team members to secure the tape measure or meter sticks against the wall. The tape measure or meter sticks should be positioned perpendicular to the floor so that the "zero end" is at 180-200 meters above the floor.

Step 2 While the tape measure or meter sticks are being positioned, the materials manager and remaining team member build the rubber band spring by looping rubber bands together until the length of the spring is about 1 meter. This task will go more quickly if each person makes about half of the spring. Then the two pieces can be joined.



Stretch bands between thumb, forefinger and middle finger.

Merge bands by grasping left band with right forefinger pulling it toward the center.



Pull both bands outward to form an intertwined loop.

Step 3 Secure one end of the rubber band spring to the pencil and the other around the neck of the bottle. Another way to secure the rubber band to the bottle is to remove the cap, insert the end of the spring in the bottle, and screw the cap back on.

Step 4 The measures manager pulls the bottle downward about 10 cm and releases it.

Step 5 The time manager starts the stopwatch when it is released and stops it at the end of 10 complete bounces. Have all team members count aloud together.

Step 6 The data manager records the number of seconds in the table under Trial 1 for 160 cm. *Hint: Starting with the spring fully extended and shortening the spring may prove more meaningful than to begin at the top and work down.*

Step 7 Repeat for Trials 2 and 3. Average the data from the 3 trials and record in the Average Time column.

Step 8 The materials manager who is holding the pencil shortens the spring by wrapping it around the pencil until the desired length of 140 is obtained.

Step 9 Continue repeating the procedure with shortened lengths of rubber band spring. Continue to record your data.

Part A: Recording the Data

1. Fill in the table with the data you collected.
   Sample response using a bottle 20 cm tall:

Spring Length (from bottom of the bottle to the pencil) ***x***	**Trial 1**	**Trial 2**	**Trial 3**	**Average Time of the Interval** ***y***
0				*Cannot be done*
20				*0*
30				*4.23*
40				*5.92*
60				*8.30*
80				*10.04*
100				*11.70*
120				*13.17*
140				*14.22*
160				*15.34*

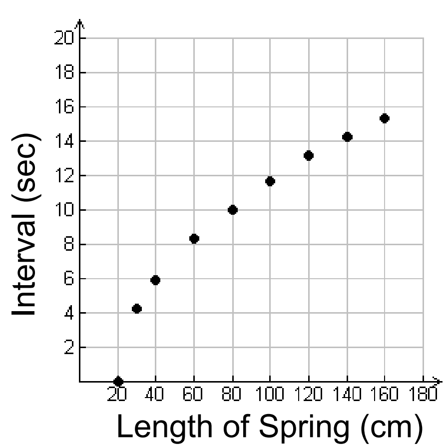
2. What is the independent variable of this situation?
   The length of the spring in centimeters

3. What is the dependent variable of this situation?
   The period of 10 bounces measured in seconds.

4. What is a reasonable domain for the set of data?
   Responses may vary. The length of the extended rubber band spring with the bottle is approximately 160 cm.

5. What is a reasonable range for the set of data?
   Responses may vary. The size and elasticity of the rubber bands used will affect the data. The maximum 10-bounce period appears to be about 16 seconds for this set of data.

6. Make a scatterplot of the data you collected.



7. Verbally conclude what happened in this data collection investigation.
   The shorter the spring, the shorter the interval appears to be. The interval time increases rapidly between intervals when the spring is at short lengths but increases more slowly as the length of the spring increases.

8. Is this data set continuous or discrete? Why?
   The data set is discontinuous; however, the data points would be closer if we were to shorten the spring by smaller increments. The data could be theoretically continuous, but it is discontinuous in any practical collection of data.

9. Does the set of data represent a function? Why?
   Yes, the data set represents a function. For each increase in the length of the spring, there is an increase in the 10-bounce interval.

10. Does the data appear to be a linear, quadratic, exponential, or some other type of parent function? Why do you think so?
    The data does not appear to be linear because it is not constant. It does not appear to be quadratic because of its behavior in relation to the y-axis, and it seems to lack a symmetric "half." It does not appear to be exponential because of its behavior in relation to the y-axis. It might be an inverse function.

Explain

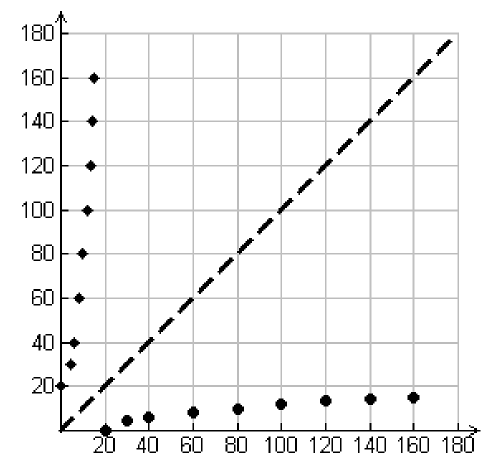
The Explain portion of the lesson is directed by the teacher to allow the students to formalize their understanding of the TEKS addressed in the lesson.

Students will work in their groups to complete the questions in **Part B** of **Bottle Bounce Activity 1.**

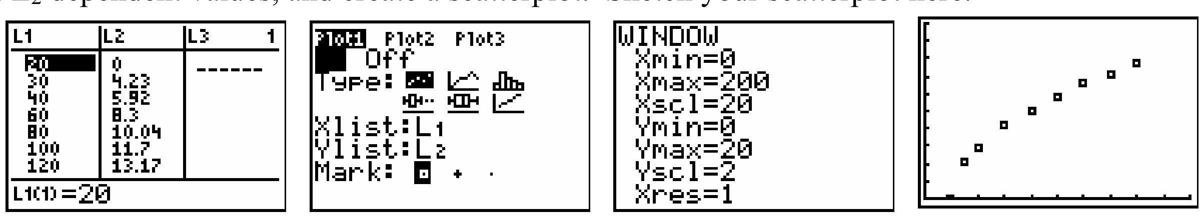
After students have completed the activity page, debrief **Part B: Interpreting the Data** of **Bottle Bounce Activity 1.** Encourage students to verbalize the connection between the graph and its equation.

Part B: Interpreting the Data

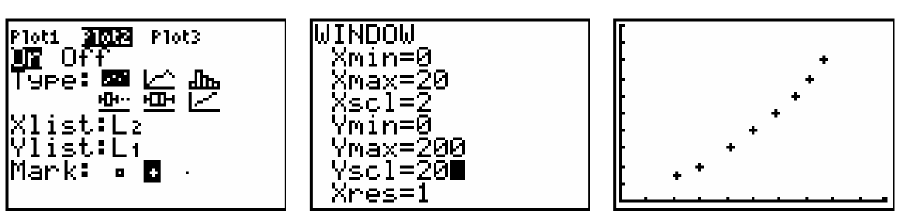
1. How could you determine whether this function is the inverse of another parent function? *Answers will vary. The data could be reflected over the line $y = x$. That is, the x values would become the y values, and vice versa. Or, the L_2 values could be mapped to L_1 using the table feature of the graphing calculator.*



2. Input your values into L_1 and L_2 of a graphing calculator, letting L_1 be independent values and L_2 dependent values, and create a scatterplot. Sketch your scatterplot here.



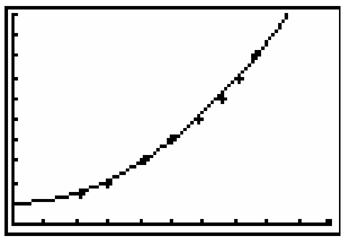
3. Create a second scatterplot using L_2 as the domain and L_1 as the range. Use a different plot symbol for this scatterplot. Determine a new domain and range, and set a new viewing window. Display the graph and sketch it here.



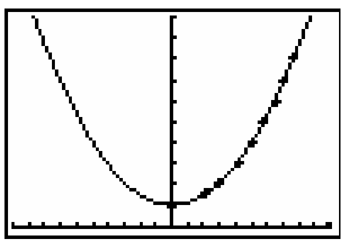
4. How do the values from the first scatterplot correspond to the values in the second scatterplot?
The x-coordinates become the y-coordinates, and the y-coordinates become the x-coordinates.

5. Which parent function do the *reflected* points most closely appear to represent?
Quadratic

6. Using what you know about transforming functions, find a function that approximates your data for Plot 2.
Answers will vary. A possible response for this data is $y = 0.6x^2 + 20$.

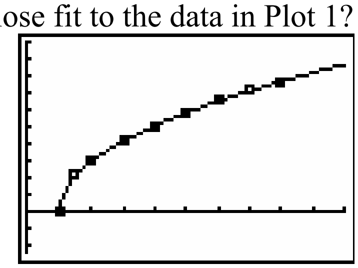


7. Does your viewing window allow you to see both sides of the parabola? If not, readjust your viewing window. *Possible viewing window*



8. How could you use this function to find a function that would approximate the first scatterplot you graphed?
I could find the inverse of the function, in this case, $y = \sqrt{\dfrac{(x-20)}{0.6}}$.

9. Reset your window to view Plot 1. Enter the equation you found in #8 in the equation editor. Is your graph a close fit to the data in Plot 1? *Responses may vary.*



10. What conclusion can you draw about the function of your bottle bounce data?
It can be represented by a square root function.

11. How are the graphs of a quadratic function and a square root function similar?
The square root function and the positive side of the quadratic function are inverses.

12. How are the graphs different?
There is no "negative" side to the square root function.

13. Why are there no negative coordinates?
Possible responses: It would not be a function. It would fail the vertical line test.

Facilitation Questions – Explore, Part B

- Why did Plot 2 reverse the domain and range of the data?
  The purpose was to find the inverse function.
- What type of function do the reflected points resemble?
  Quadratic (half of a sideways parabola)

Elaborate

The Elaborate portion of the lesson provides an opportunity for the student to apply the concepts of the TEKS within a new situation. Students will extend their understanding of square root functions as they solve square root equations. This part of the lesson is designed for a combination of whole class and small group format.

1. Distribute **Graph Match** to students. Ask them to follow directions on the activity.
2. Debrief **Graph Match**.

Facilitation Questions – Elaborate

- How does the graph of the reflected (quadratic) data compare to the graph of $y = x^2$?
  It is wider (vertical compression) and is translated up.
- How will that help you decide how to transform the function to better fit the data?
  The coefficient of x^2 (the "a" value) should be greater than zero, but less than one. There should be a "c" value that is positive.
- What happens to the graph if a number is added or subtracted from the x?
  The graph will translate (shift) left or right (horizontally).
- What happens to the graph if there is a negative sign in front of the radical?
  The parent function is reflected over the x-axis.
- Is it possible to take the square root of –x ?
  Yes, if the x values are negative. It becomes a reflection of the parent function over the y-axis.

3. Distribute **Representing Square Root Functions Activity 3**. Explain to students that part of the information in the table (first page of activity) is missing. They will work to complete it.
4. Do one function as a whole class in order to formalize the vocabulary of the various transformations and the idea of domain and range for square root functions.
5. Students could then work in groups to complete the table, OR you could assign each group only one of the functions. Then each group could display their one function in its various representations on chart paper.
6. The second page of the activity is the recording sheet for the Card Sort. To make the card sort, make copies of the **Answer Key** page and cut out each entry. Put them in a small bag or envelope.
7. Give each group a set of cards. They will use the cards to decide what functions to write for each entry on the recording chart. This activity will help them cement the idea of square root transformations and lead them to the symbolic notation. Display the chart (**Transparency #4**) and have student volunteers record and explain their results.

Graph Match

Input the following functions into the equation editor of your graphing calculator. You may use the calculator graph to help you determine which graph is the best match. All graphs were drawn using a standard viewing window.

D 1. $y = \sqrt{x}$

A 2. $y = \sqrt{x-5}$

B 3. $y = \sqrt{x+3}$

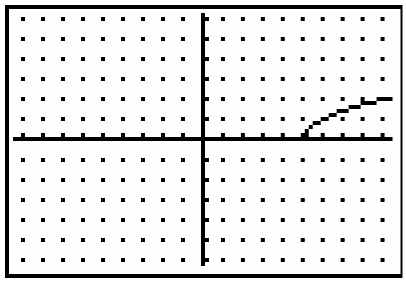
C 4. $y = \sqrt{x} - 6$

F 5. $y = \sqrt{x} + 4$

H 6. $y = 3\sqrt{x}$

E 7. $y = \frac{1}{2}\sqrt{x}$

G 8. $y = -\sqrt{x}$

A.


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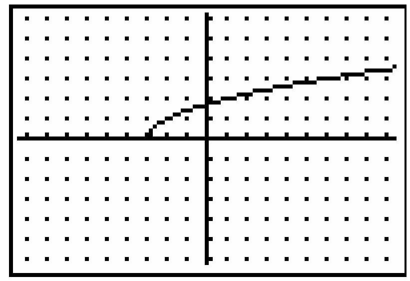

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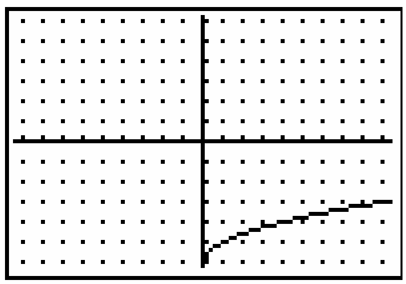

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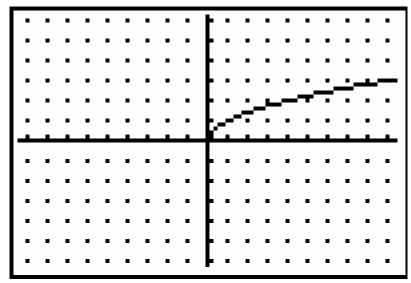

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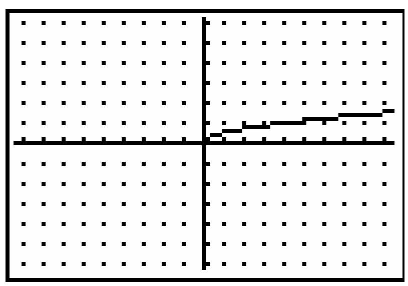

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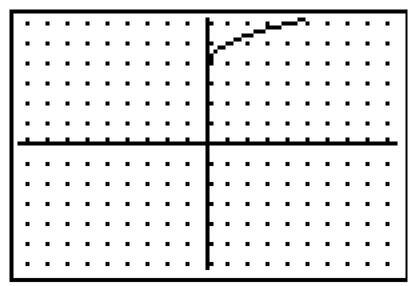

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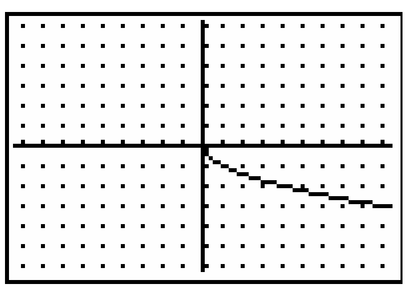

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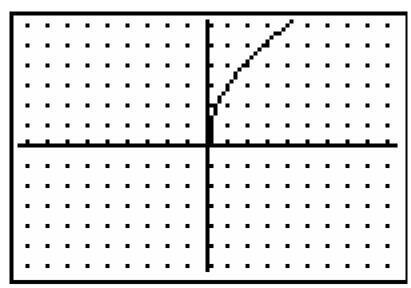
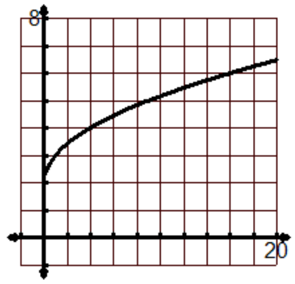


Representing Square Root Functions Activity 3

Answer Key

Complete the table.

<table>
<tr><th>Function</th><th>Table of Values</th><th>Graph</th><th>Domain and Range</th><th>Transformation(s) (compared to graph of $y=\sqrt{x}$)</th></tr>
<tr><td>$y=\sqrt{x}+2$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>2</td></tr><tr><td>1</td><td>3</td></tr><tr><td>4</td><td>4</td></tr><tr><td>9</td><td>5</td></tr><tr><td>16</td><td>6</td></tr></table></td><td></td><td>$x\ge 0$
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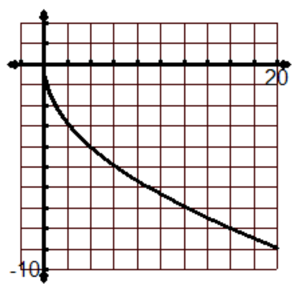
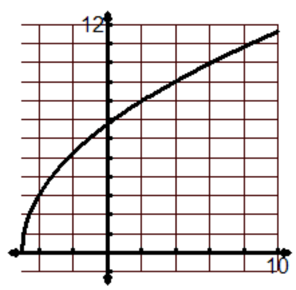
<tr><td>$y=-2\sqrt{x}$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>–2</td></tr><tr><td>4</td><td>–4</td></tr><tr><td>9</td><td>–6</td></tr><tr><td>16</td><td>–8</td></tr></table></td><td></td><td>$x\ge 0$
$y\le 0$</td><td>• Vertical stretch by a factor of 2
• Reflection over the x-axis</td></tr>
<tr><td>$f(x)=3\sqrt{x+5}$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>–5</td><td>0</td></tr><tr><td>–4</td><td>3</td></tr><tr><td>–1</td><td>6</td></tr><tr><td>4</td><td>9</td></tr><tr><td>11</td><td>12</td></tr></table></td><td></td><td>$x\ge -5$
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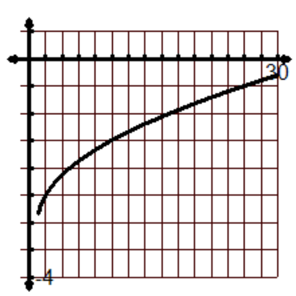
<tr><td>$y=\frac{1}{2}\sqrt{x-1}-3$</td><td><table><tr><th>$x$</th><th>$y$</th></tr><tr><td>1</td><td>–3</td></tr><tr><td>5</td><td>–2</td></tr><tr><td>17</td><td>–1</td></tr><tr><td>37</td><td>0</td></tr><tr><td>65</td><td>1</td></tr></table></td><td></td><td>$x\ge 1$
$y\ge -3$</td><td>• Vertical compression by a factor of $\frac{1}{2}$
• Horizontal translation right 1
• Vertical translation down 3</td></tr>
</table>

Card Sort

1. Place the cards in the proper row and column.

Description	Example	Example	Notation
Vertical Translation Up	$y=\sqrt{x}+5$	$y=\sqrt{x}+1$	$y=\sqrt{x}+k$ $k>0$
Vertical Translation Down	$y=\sqrt{x}+-3$	$y=\sqrt{x}-5$	$y=\sqrt{x}+k$ $k<0$
Horizontal Translation Left	$y=\sqrt{x--4}$	$y=\sqrt{x+5}$	$y=\sqrt{x-h}$ $h<0$
Horizontal Translation Right	$y=\sqrt{x-+6}$	$y=\sqrt{x-5}$	$y=\sqrt{x-h}$ $h>0$
Vertical Stretch	$y=3\sqrt{x}$	$y=5\sqrt{x}$	$y=a\sqrt{x}$ $a>1$
Vertical Compression	$y=\frac{2}{3}\sqrt{x}$	$y=\frac{1}{5}\sqrt{x}$	$y=a\sqrt{x}$ $0<a<1$
Reflection	$y=-\sqrt{x}$	$y=-5\sqrt{x}$	$y=a\sqrt{x}$ $a<0$

2. Describe the role of *a*.
   $|a|$ *is the vertical stretch/compression factor. If a<0, the graph is reflected across the x-axis.*

3. Describe the role of *h*.
   h is the horizontal translation.

4. Describe the role of *k*.
   k is the vertical translation.

5. Using *x*, *a*, *h*, and *k*, write an equation that could be used to summarize the transformations to the square root function.
   $y = a\sqrt{x-h}+k$ or $f(x) = a\sqrt{x-h}+k$

Evaluate

The Evaluate portion of the lesson provides the student with an opportunity to demonstrate his or her understanding of the TEKS addressed in the lesson. This part of the lesson is designed for small groups.

1. Provide each group with the **Performance Assessment: The Pirate Ship** activity sheet.
2. Upon completion of the activity sheet, a rubric should be used to assess student understanding of the concepts addressed in the lesson.

Answers and Error Analysis for selected response questions:

Question Number	TEKS	Correct Answer	Conceptual Error	Conceptual Error	Procedural Error	Procedural Error	Guess
1	2A.9D	D	A	B	C		
2	2A.9A	D	B	C			A
3	2A.9B	B	A	D	C		
4	2A.9C	C	D		A	B	

The Pirate Ship

Answer Key

At Thalia's favorite amusement park, there is a ride called the "Pirate Ship." People sit in what looks like a huge ship. The "ship" then swings back and forth, moving like a pendulum. Thalia notices that it takes somewhere between 7 and 8 seconds for the ride to make one complete swing back and forth.

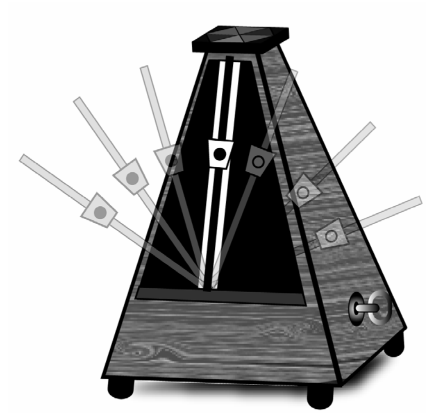
> The function that represents the time in seconds of one complete swing, *t*, based on the height of the pendulum arm, *h*, in feet, is $t = 2\pi\sqrt{\frac{h}{32}}$.

What would be the possible domain of values for the height of the "Pirate Ship" ride, if the time for one complete swing is between 7 and 8 seconds?

Solution: The height is between (approximately) 39.8 and 51.9 feet.

Transparency: Metronome

A *metronome* is frequently used in music to mark exact time using a repeated tick.



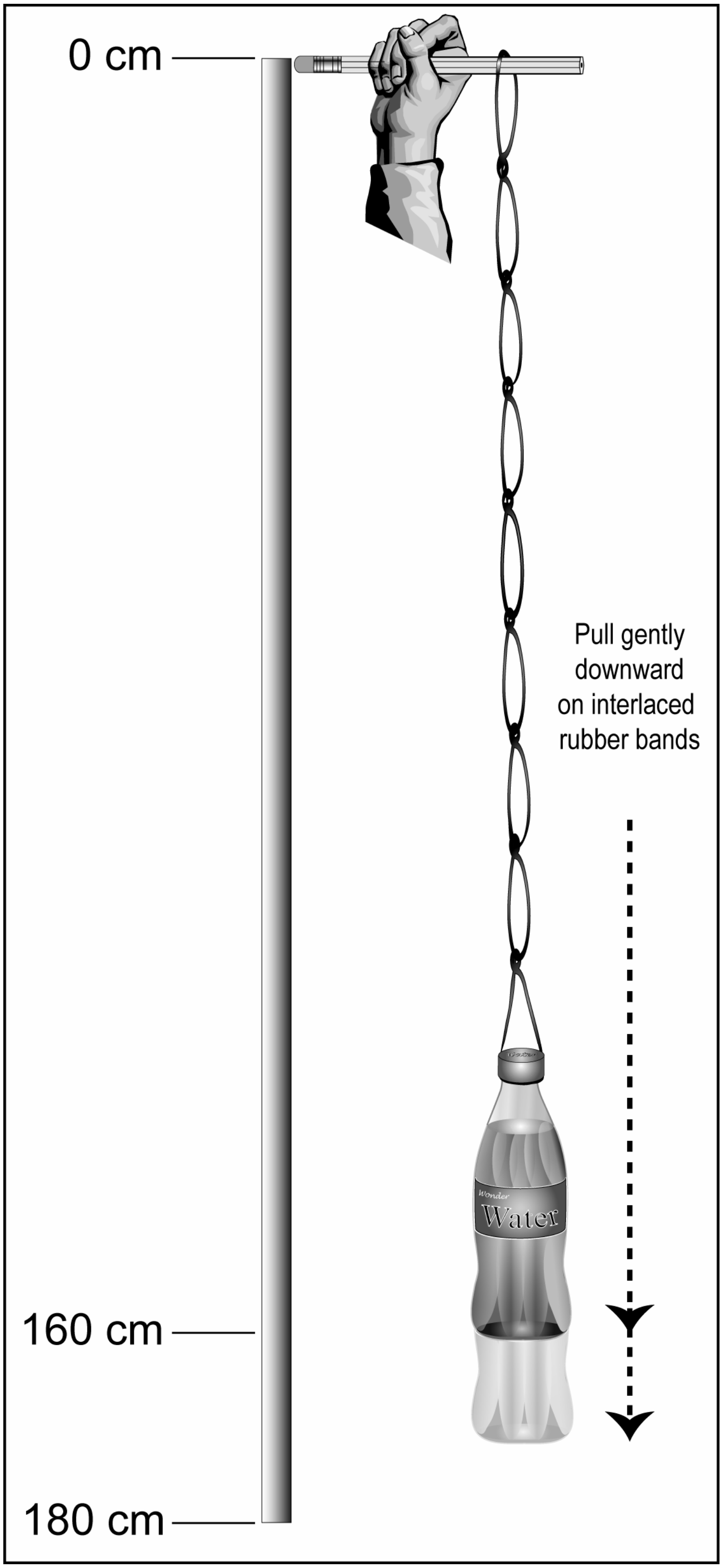
Musicians, choirs, bands, and orchestras all use metronomes to ensure that the beat of the music is consistent with the instructions of the composer and does not unintentionally speed up or slow down while the piece is being played.

How does the metronome keep a slow tempo or a fast tempo?

How do they compare?

Transparency

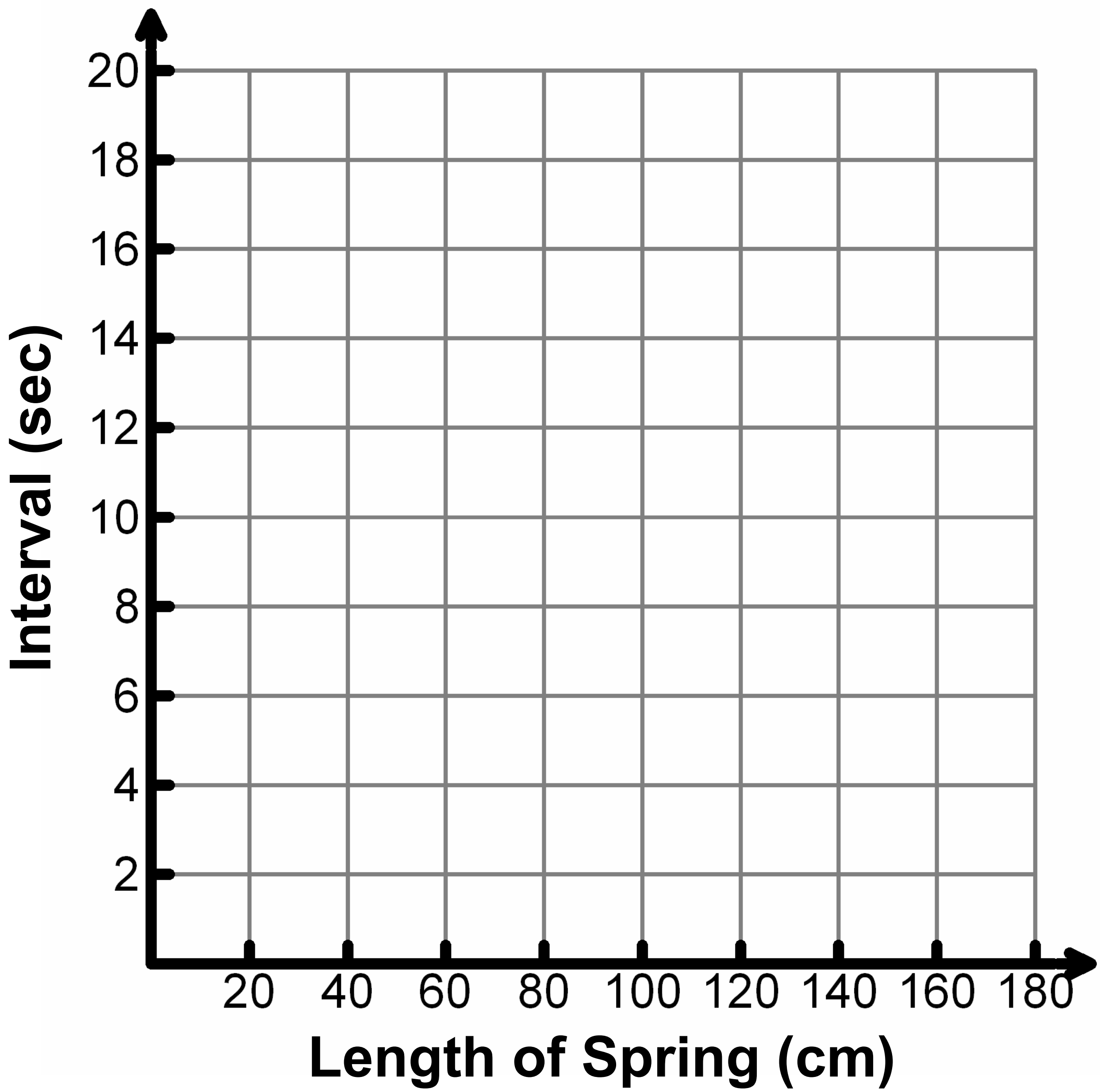
Bottle Bounce: Activity 1



Transparency: Recording the Data

Spring Length (from bottom of the bottle to the pencil) *(x)*	**Trial 1**	**Trial 2**	**Trial 3**	**Average Time of the Interval** *(y)*
0				
20				
30				
40				
60				
80				
100				
120				
140				
160				

Transparency: Recording the Data

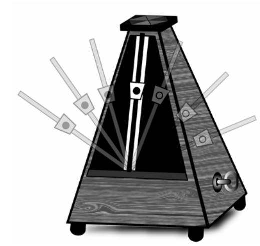


Card Sort

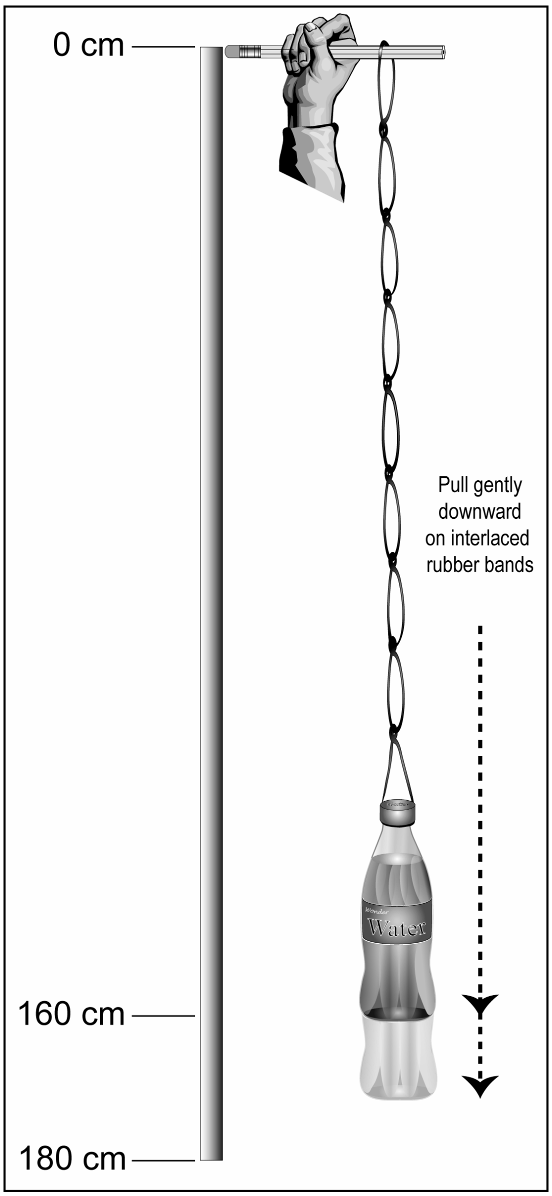
Description	Example	Example	Notation
Vertical Translation Up			
Vertical Translation Down			
Horizontal Translation Left			
Horizontal Translation Right			
Vertical Stretch			
Vertical Compression			
Reflection			

Bottle Bounce
Activity 1

A metronome is frequently used in music to mark exact time using a repeated tick. Individual instrumentalists, choirs, bands, and orchestras all use metronomes to ensure that the beat of the music is consistent with the instructions of the composer and does not unintentionally speed up or slow down while the piece is being played.



The data collection activity that follows will model the movement of the metronome by letting a filled bottle bounce up and down.



A filled 16-ounce bottle will be suspended from a pencil by a 1-meter strand of rubber bands. When the bottle is attached the stretched rubber band spring should extend so that the bottom of the bottle of water is at least 140 cm below the pencil.

One person will hold the pencil (eraser end held firmly against the wall) at a height of 160 to 200 centimeters above the floor. The rubber band spring should be attached close to the opposite end of the pencil so the bottle can bounce without striking the wall. To simulate moving the weight up or down on the metronome, the rubber band spring will be wrapped around the pencil until the bottom of the bottle is raised to the desired height.

Collecting Data
In this investigation, a metric tape measure should be taped to the wall. You will suspend a filled 16-ounce bottle by a 1-meter rubber band spring from a pencil firmly held perpendicular to a wall at a height of 160-200 centimeters. The difference between the length of the rubber band spring and the suspension height should allow the bottle to bounce without touching the floor.

The length of the rubber band spring will be shortened by wrapping the spring around the pencil so that data is collected at 20-centimeter intervals.

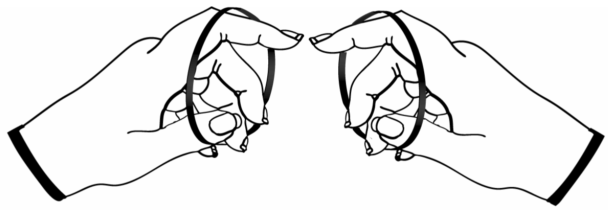
You will be working in groups of 4. Each person in the group has a job.

Materials manager:	Gets the necessary materials, directs the team in setting up the investigation, holds the pencil with the suspended bottle, and shortens the spring when needed.
Measure manager:	Measures the distance from the pencil to the bottom of the bottle for each length, initiates the bounce by pulling the suspended bottle down an additional 10 centimeters and counts the bounces (10 at each height).
Time manager:	Uses a stop watch to determine the length of each 10-bounce period of time. The time starts when the bottle is released by the measures manager and ends when the bottle completes its 10th bounce.
Data manager:	Records the necessary measurements in the table and shares the data with the team.

Set-up Instructions

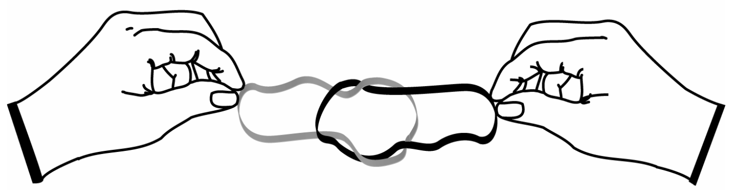
Step 1 The materials manager should get the necessary materials and ask two of the team members to secure the tape measure or meter sticks against the wall. The tape measure or meter sticks should be positioned perpendicular to the floor so that the “zero end” is at 180-200 centimeters above the floor.

Step 2 While the tape measure or meter sticks are being positioned, the materials manager and remaining team member build the rubber band spring by looping rubber bands together until the length of the spring is about 1 meter. This task will go more quickly if each person makes about half of the spring. Then the two pieces can be joined.



Stretch bands between thumb, forefinger and middle finger.

Merge bands by grasping left band with right forefinger pulling it toward the center.



Pull both bands outward to form an intertwined loop.

Step 3 Secure one end of the rubber band spring to the pencil and the other around the neck of the bottle. Another way to secure the rubber band to the bottle is to remove the cap, insert the end of the spring in the bottle, and screw the cap back on.

Step 4 The measures manager secures the spring so that it is approximately 160 centimeters in length. After the bottle remains motionless for a few seconds, he should measure the actual length of the spring. The length of the spring includes the length of the rubber band and the length of the bottle.

Step 5 The measures manager then pulls the bottle downward about 10 cm and releases it.

Step 6 The time manager starts the stopwatch when it is released and stops it at the end of 10 complete bounces. Have all team members count aloud together.

Step 7 The data manager records the number of seconds in the table under Trial 1 for 160 cm. *Hint: Starting with the spring fully extended and shortening the spring may prove more meaningful than to begin at the top and work down.*

Step 8 Repeat for Trials 2 and 3. Average the data from the 3 trials and record in the Average Time column.

Step 9 The materials manager who is holding the pencil shortens the spring by wrapping it around the pencil until the desired length of 140 is obtained.

Step 10 Continue repeating the procedure with shortened lengths of rubber band spring. Continue to record your data.

Part A: Recording the Data

1. Fill in the table with the data you collected.

Approximate Spring Length (from bottom of the bottle to the pencil in cm)	Actual Length of Spring (cm) *x*	Trial 1	Trial 2	Trial 3	Average Time of the Interval (sec) *y*
0					
20					
30					
40					
60					
80					
100					
120					
140					
160					

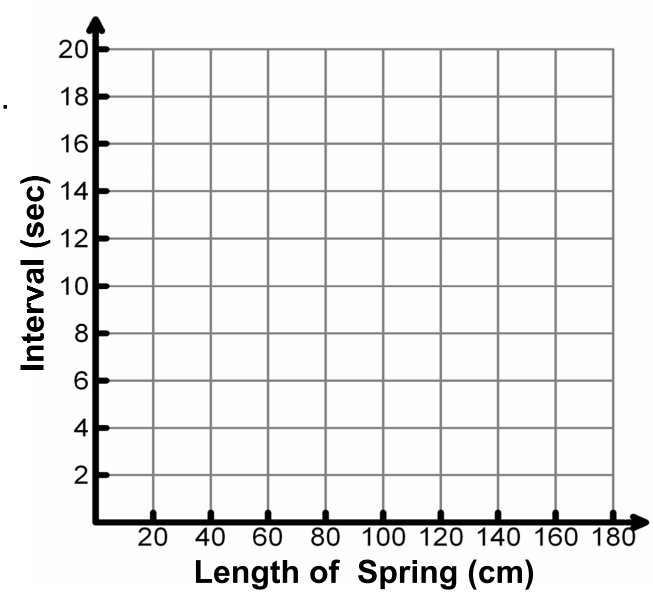
2. What is the independent variable of this situation?

3. What is the dependent variable of this situation?

4. What is a reasonable domain for the set of data?

5. What is a reasonable range for the set of data?

6. Make a scatterplot of the data you collected.



7. Verbally conclude what happened in this data collection investigation.

8. Is this data set continuous or discrete? Why?

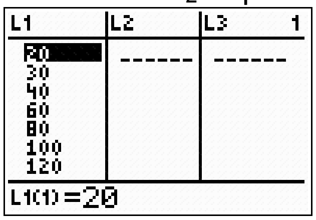
9. Does the set of data represent a function? Why?

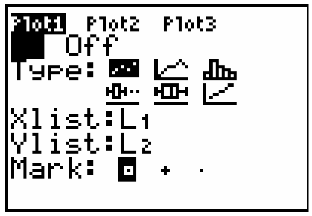
10. Does the data appear to be a linear, quadratic, exponential, or some other type of parent function? Why do you think so?

Part B: Interpreting the Data

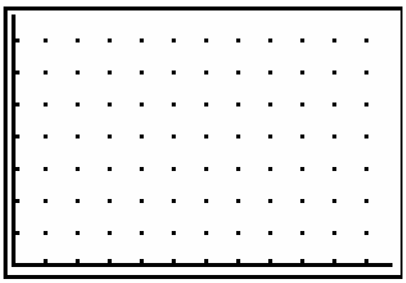
1. How could you determine whether this function is the inverse of another parent function?

2. Input your values into L_1 and L_2 of a graphing calculator, letting L_1 be independent values and L_2 dependent values, and create a scatterplot.

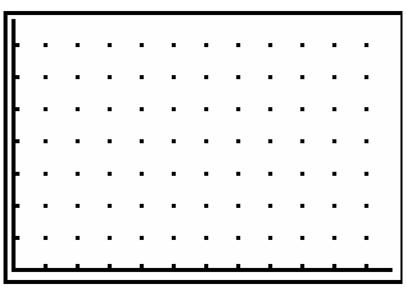




Sketch your scatterplot here.



3. Create a second scatterplot using L_2 as the domain and L_1 as the range. Use a different plot symbol for this scatterplot. Determine a new domain and range, and set a new viewing window. Display the graph and sketch it here.



4. How do the values from the first scatterplot correspond to the values in the second scatterplot?

5. Which parent function do the *reflected* points most closely appear to represent?

6. Using what you know about transforming functions, find a function that approximates your data for Plot 2.

7. Does your viewing window allow you to see both sides of the parabola? If not, readjust your viewing window.

8. How could you use this function to find a function that would approximate the first scatterplot you graphed?

9. Reset your window to view Plot 1. Enter the equation you found in #8 in the equation editor. Is your graph a close fit to the data in Plot 1?

10. What conclusion can you draw about the function of your bottle bounce data?

11. How are the graphs of a quadratic function and a square root function similar?

12. How are the graphs different?

13. Why are there no negative coordinates?

Graph Match

Input the following functions into the equation editor of your graphing calculator. You may use the calculator graph to help you determine which graph is the best match. All graphs were drawn using a standard viewing window.

_____ 1. $y = \sqrt{x}$

_____ 2. $y = \sqrt{x-5}$

_____ 3. $y = \sqrt{x+3}$

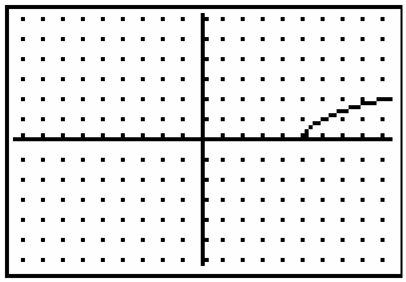
_____ 4. $y = \sqrt{x} - 6$

_____ 5. $y = \sqrt{x} + 4$

_____ 6. $y = 5\sqrt{x}$

_____ 7. $y = \dfrac{1}{2}\sqrt{x}$

_____ 8. $y = -\sqrt{x}$

A.


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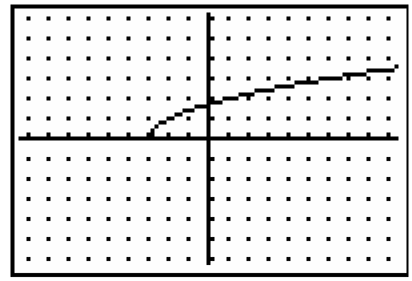

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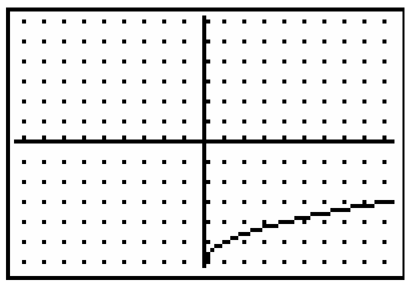

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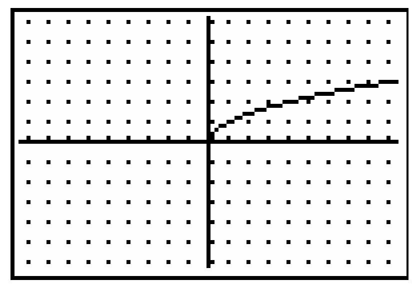

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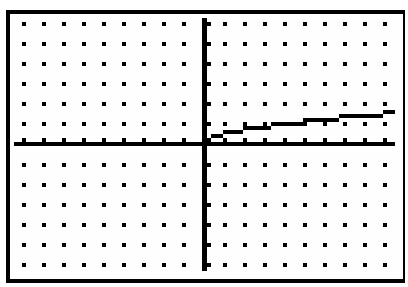

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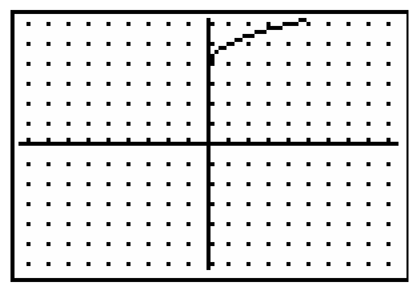

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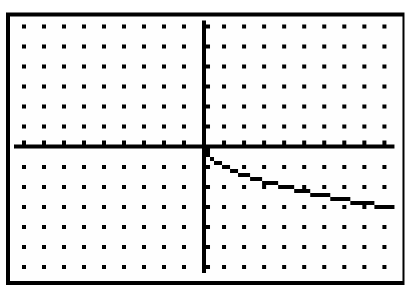

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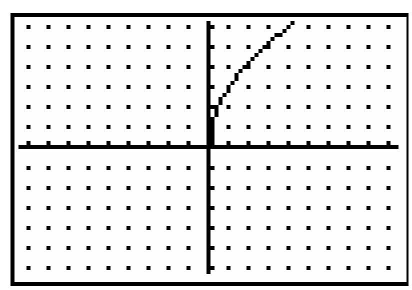

Representing Square Root Functions
Activity 3

Complete the table.

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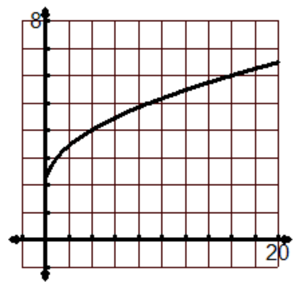
<tr><td></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>4</td><td></td></tr><tr><td>9</td><td></td></tr><tr><td>16</td><td></td></tr></table></td><td>
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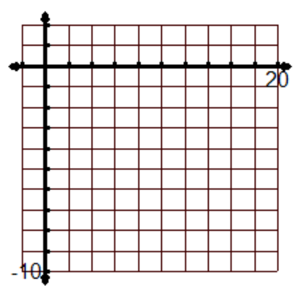
<tr><td>$y = -2\sqrt{x}$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr></table></td><td>
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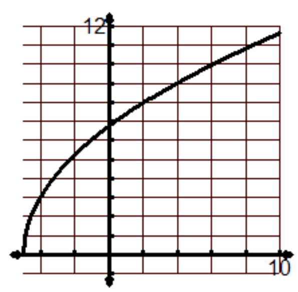
<tr><td></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>–5</td><td>0</td></tr><tr><td>–4</td><td>3</td></tr><tr><td>–1</td><td>6</td></tr><tr><td>4</td><td>9</td></tr><tr><td>11</td><td>12</td></tr></table></td><td>
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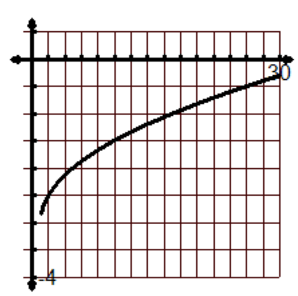
<tr><td>$y = \frac{1}{2}\sqrt{x-1} - 3$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>1</td><td>–3</td></tr><tr><td>5</td><td>–2</td></tr><tr><td>17</td><td>–1</td></tr><tr><td>37</td><td>0</td></tr><tr><td>65</td><td>1</td></tr></table></td><td>
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</table>

Card Sort

1. Place the cards in the proper row and column.

Description	Example	Example	Notation
Vertical Translation Up			
Vertical Translation Down			
Horizontal Translation Left			
Horizontal Translation Right			
Vertical Stretch			
Vertical Compression			
Reflection			

2. Describe the role of *a* in a square root function.

3. Describe the role of *h* in a square root function.

4. Describe the role of *k* in a square root function

5. Using *x, a, h,* and *k,* write an equation that could be used to summarize the transformations to the square root function.

The Pirate Ship

At Thalia's favorite amusement park, there is a ride called the "Pirate Ship." People sit in what looks like a huge ship. The "ship" then swings back and forth, moving like a pendulum. Thalia notices that it takes somewhere between 7 and 8 seconds for the ride to make one complete swing back and forth.

The function that represents the time in seconds of one complete swing, *t*, based on the height of the pendulum arm, *h*, in feet, is $t = 2\pi\sqrt{\frac{h}{32}}$.

What would be the possible domain of values for the height of the "Pirate Ship" ride, if the time for one complete swing is between 7 and 8 seconds?

Symbolic	Table	Graph

Student Lesson: Absolute Value Functions

TEKS:

a(5) Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a(6) Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.1 **Foundations for functions**. The student uses properties and attributes of functions and applies functions to problem situations.

2A.1A The student is expected to identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

2A.1B The student is expected to collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

2A.2 **Foundations for functions.** The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

2A.2A The student is expected to use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

2A.4 **Algebra and geometry.** The student connects algebraic and geometric representations of functions.

2A.4A The student is expected to identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x (($f(x) = \sqrt{x}$), and reciprocal of $x\left(f(x) = \dfrac{1}{x}\right)$.

2A.4B The student is expected to extend parent functions with parameters such as a in $f(x) = a/x$ and describes the effects of the parameter changes on the graph of parent functions.

Objectives:

At the end of this student lesson, students will be able to:

- describe the absolute value parent function as a pair of linear functions with restricted domain and
- identify, sketch, and describe the effects of parameter changes on the graph of the absolute value parent function.

TAKS™ Objectives Supported:

While the Algebra II TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials:

Prepare in Advance: Copies of participant pages, copies of **The Fire Station Problem Graphic Sheets** taped together

Presenter Materials: **Overhead graphing calculator**

Per group: **The Fire Station Problem Graphic Sheets** taped together to represent a street, chart paper

Per participant: Copy of participant pages, graphing calculator

Engage

The Engage portion of the lesson is designed to provide students with a concrete connection to the Explore phase of the lesson.

Facilitation Questions:

- There are many factors that affect the price of home insurance. "What do you think some of the factors might be?"
  Possible responses may include: the age of the home, the crime rate in the neighborhood, the type of construction of the home, weather in the area, such as hurricanes, tornadoes, flooding, a flood plain location, smoke alarm and security system, quality of the fire department, quality of the police department, size of the home, value of the home, working fire hydrants, location of the fire department, or distance to the nearest fire hydrant.

- If students do not mention the distance to a fire hydrant or fire station, ask,
  "Do you think that the distance from your home to a fire station or fire hydrant would matter?"
  Hopefully, students will say that the distance does matter.

- "Why do you think it matters?"
  Possible responses: The distance your home is from a fire station or fire hydrant may affect the fire department's response time. If the fire trucks can get water on the home faster, there may be less damage for the insurance company to pay.

- How far do you think your home is from the fire station?
  Responses may vary.

Explore

The Explore portion of the lesson provides the student with an opportunity to explore concretely the concept of absolute value functions.

At the end of the Explore phase, students should be able to describe an absolute value function as two linear functions.

1. Distribute **The Fire Station Problem** and the graphics sheets.

2. Ask students to tape the graphics sheets together if you have not already done so.

3. Encourage students to answer the questions on **The Fire Station Problem** activity sheet.

The Fire Station Problem

Answer Key

A fire station is located on Main Street and has buildings at every block to the right and to the left. You will investigate the relationship between the address number on a building and its distance in blocks from the fire station.

1. Complete the table below that relates the address of a building (*x*) with its distance in blocks from the fire station (*y*).

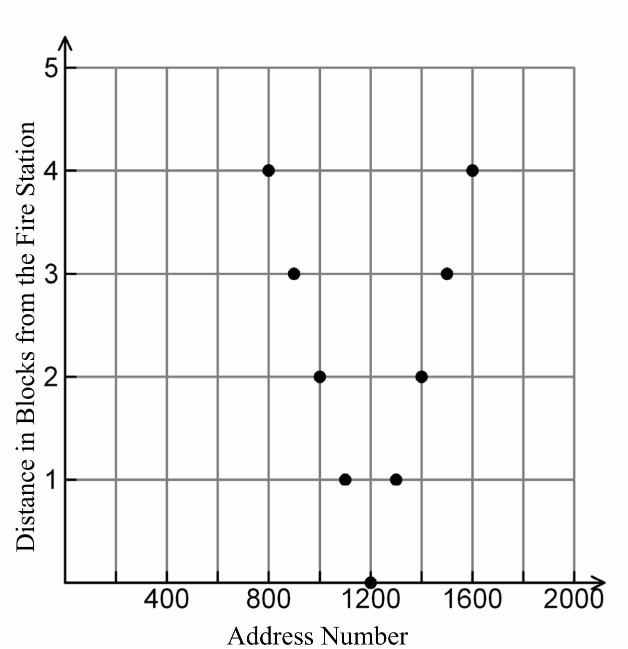
Address Number (*x*)	Distance in Blocks from the Fire Station (*y*)
800	*4*
900	*3*
1000	*2*
1100	*1*
1200	*0*
1300	*1*
1400	*2*
1500	*3*
1600	*4*

2. Which building is 2 blocks away from the fire station? Explain your answer.
There are two buildings that are 2 blocks from the fire station: the church and the ice cream shop. The church is 2 blocks to the left of the fire station (as you look towards the fire station), and the ice cream shop is 2 blocks to the right of the fire station.

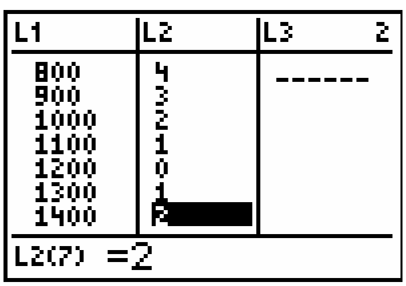
3. If we send someone to the building that is 2 blocks away from the fire station, how will she know that she has arrived at the correct place?
She will not know she has arrived at the correct place unless we tell her which direction to walk, or we indicate whether the building is to the right or the left of the fire station.

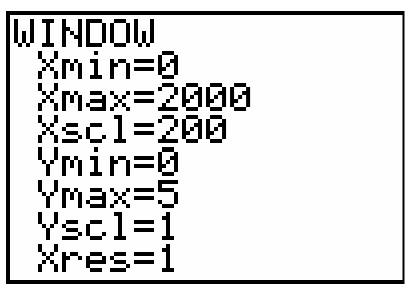
4. How do we describe two numbers that represent the same distance from a location?
We describe them in terms of direction (north, south, east, west, 12 o'clock, 1 o'clock, etc.).

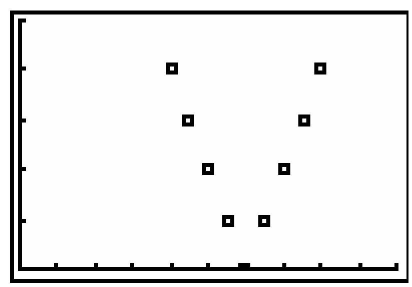
Draw a scatterplot that represents the data in the table.



5. **Make a scatterplot of your data using your graphing calculator. Describe your viewing window.**
   Responses may vary. Possible answers are shown below.







6. **What function or functions might you use to describe the scatterplot?**
   Responses may vary. Two linear functions used together might describe the data. A quadratic function does not describe the data.

7. **Find two linear functions that pass through the data points. What process did you use to find the equations of the lines?**
   The linear functions $y = –0.01x+12$ and $y = 0.01x – 12$ model the set of data. Students may use the point-slope formula, the slope-intercept formula or the table.

Work backward in the table from (1200, 0) to the point (0, b) on the positive y-axis, with the rate of change: m = –1/100 or –0.01. The y-intercept is 12. So, one of the linear equations is y = –0.01x + 12.

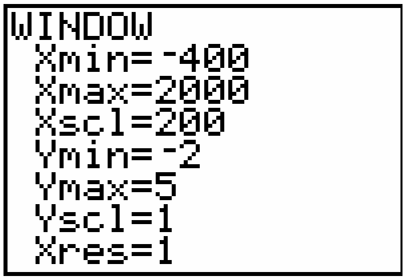
x	*y*
0	12
100	11
200	10
300	9
400	8
500	7
600	6
700	5
800	4
900	3
1000	2
1100	1
1200	0

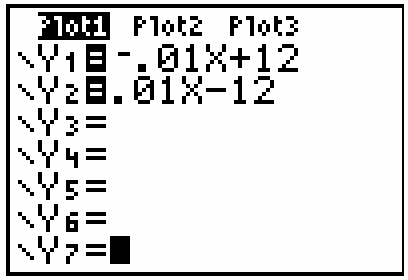
Work backward in the table from (1200, 0) to the point (0, b) on the negative y-axis, with the rate of change: m = 1/100 or 0.01. The y-intercept is –12. So, one of the linear equations is y = 0.01x – 12.

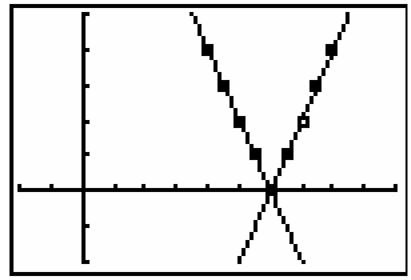
x	*y*
0	–12
100	–11
200	–10
300	–9
400	–8
500	–7
600	–6
700	–5
800	–4
900	–3
1000	–2
1100	–1
1200	0

8. Graph the equations on your calculator. How are the equations similar? How are they different?
The equations are similar because they each have a constant rate of change. The absolute value of the slopes of the equations is 0.01. The absolute value of the y-intercepts of the equations is 12. The equations are different because the slope of one equation is –0.01 and the slope of the other is 0.01. The y-intercept of one equation is 12 and the y-intercept of the other is –12.

9. If necessary adjust the window to clearly see the intersection of the two lines. What does the intersection of these two lines represent?







The intersection of the two lines represents the location of the fire station.

10. Where do the equations fit the graph of the data points? Where do the equations not fit the graph of the data points?
The equations fit the data points above the x-axis. The equations do not fit the data points below the y-axis. For x values larger than 1200 the equation $y = 0.01x - 12$ fits. For x values smaller than 1200 the equation $y = -0.01x + 12$ fits.

11. How well do the linear functions model the data points?
Linear functions model the data points above the point of intersection well. Below the point of intersection, the functions do not fit the data.

Explain

The Explain portion of the lesson is directed by the teacher to allow the students to formalize their understanding of the TEKS addressed in the lesson. Use the questions to elicit student groups to share their responses.

At the end of the Explain phase, students should be able to communicate the characteristics of the absolute value parent function.

1. **Distribute Part 2: The Fire Station Problem to students. Ask them to work through the activity. Actively monitor student groups, providing assistance as necessary.**

2. **Debrief Part 2: The Fire Station Problem.**

3. **Use a vocabulary organizer to describe and define the absolute value of a number and the absolute value parent function.**

4. **Formalize:**

   The absolute value of a number is defined as $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$.

   The absolute value parent function is defined as $f(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$.

Part 2: The Fire Station Problem

Answer Key

1. **What is the domain and range of each of the linear functions that model the fire station problem?**
   For both linear functions, the domain is all real numbers. The range is also all real numbers.

2. **How do these domains and ranges compare to those of the data set?**
   For the data set, the domain is {800, 900, 1000, 1100, 1200, 1300, 1400, 1500, 1600, 1700, 1800} and the range is {0, 1, 2, 3, 4}.

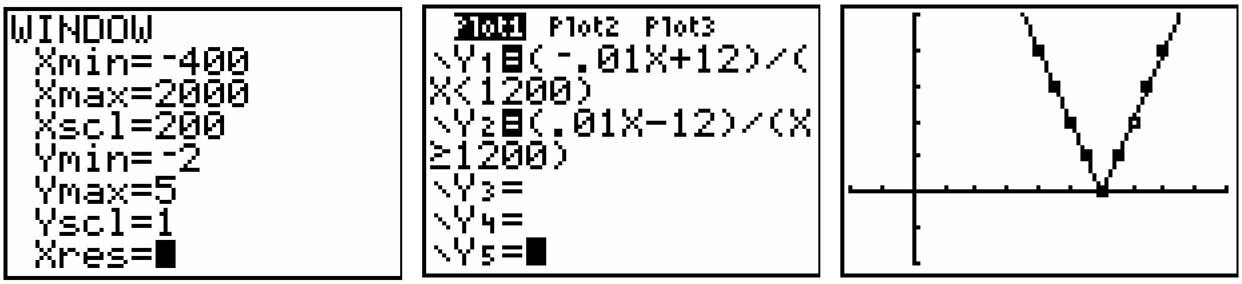
   The function y = –0.01x + 12 that models the left side of the data contains the data points (800, 4), (900, 3), (1000, 2), (1100, 1), and (1200, 0). The function y = 0.01x – 12 that models the right side of the data contains the data points (1200, 0), (1300, 1), (1400, 2), (1500, 3), and (1600, 4).

3. **How do the equations compare?**
   The equations are opposites of each other.
   One of the equations is y = 0.01(x – 1200) or, y = 0.01x –12; the other equation is y = –0.01(x – 1200) or, y = –0.01x + 12.

4. **At what point do the graphs of the equations intersect?**
   The lines intersect at (1200, 0).

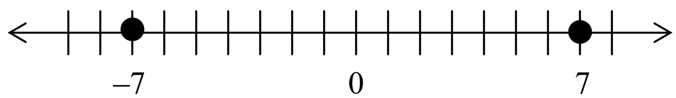
5. **Which parts of the graphs of the lines model our data set? Which parts do not? Why?**
   The graphs of the lines above the x-axis model our data points. The parts of the graphs below the x-axis do not model our data points. The lines that model our data have restricted domains.

6. **What happens when you graph the linear functions over the given domain? Sketch and describe your resulting graph. Why do you think this happens?**



   The inequalities x < 1200 and x ≥ 1200 are Boolean operators that return values of 1 when they are true and 0 when they are false. When the inequalities are false, the denominators of Y1 or Y2 become 0. Since you cannot divide by 0, there is no point on the graph to plot. Thus, the inequalities in the denominator help us to restrict graphically the domain of the function.

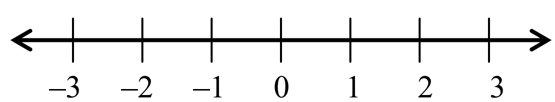
Numerically, when we consider the distance of a number from 0 on the number line, we call that distance the *absolute value* of the number.



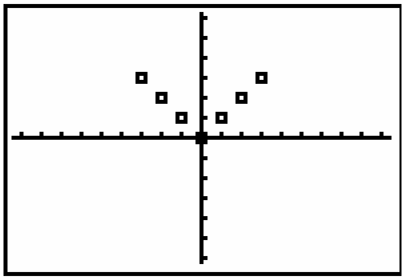
For example, $|-7| = 7$, since –7 is 7 units to the left of 0. Similarly, $|7| = 7$, since 7 is 7 units to the right of 0.

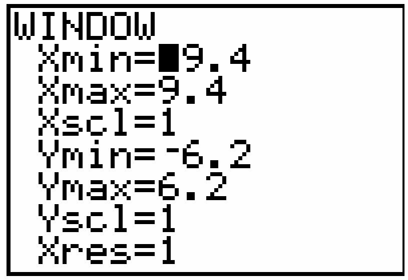
7. **The *x*-coordinates below represent locations on a number line. The *y*-coordinates represent the distance that location is from 0. For the given *x*-values, use the number line to find the corresponding *y*-values.**

x	*y*
–3	*3*
–2	*2*
–1	*1*
0	*0*
1	*1*
2	*2*
3	*3*



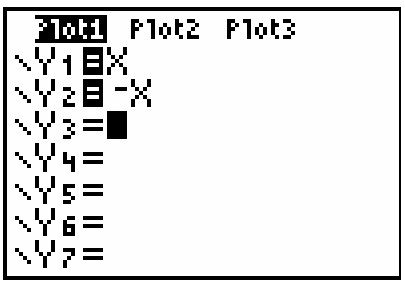
8. Make a scatterplot of *y* versus *x* on your graphing calculator. Describe your window. Sketch your scatterplot.

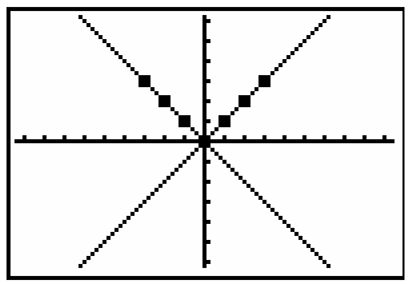




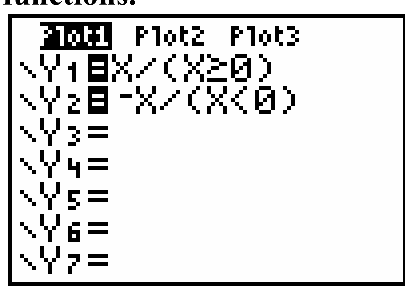
9. What is the shape of your scatterplot?
The shape is a "v."

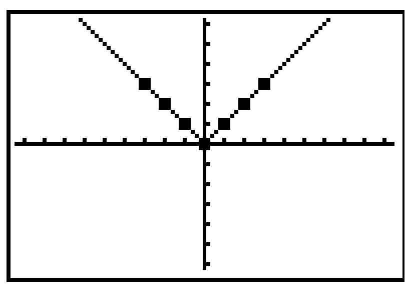
10. Find two linear functions that model this situation. Graph these functions.
$y_1 = x$ *and* $y_2 = -x$



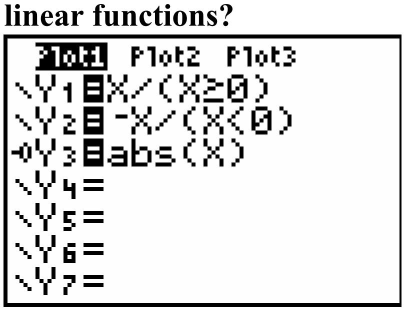


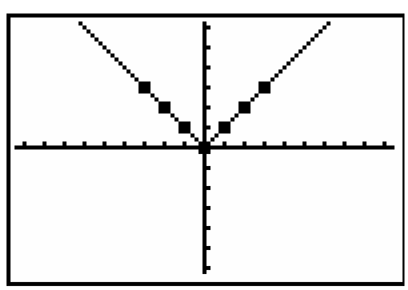
11. Restrict the domain so that the model is an even better fit. Graph the restricted functions.





12. Now try $y_3 = |x|$ using the trace bubble. How does this function compare to the two linear functions?





The equation y_3 *has the same graph as the restricted linear functions.*

13. This is a new parent function to add to your list of parent functions. With which other parent functions are you familiar?
Responses may vary. The linear parent function, $y = x$, and the quadratic parent function, $y = x^2$, are two possible parent functions with which students may be familiar.

14. What is the parent function for absolute value?
The parent function is $f(x) = |x|$.

15. What are the characteristics of absolute value functions?
Absolute value functions graph in the shape of a "v." The slope of the left side of the graph is the opposite of the slope of the right side of the graph. The graph has line symmetry. The absolute value parent function is produced by reflecting the part of the graph of $y = x$ to the left of $x = 0$ above the x-axis.

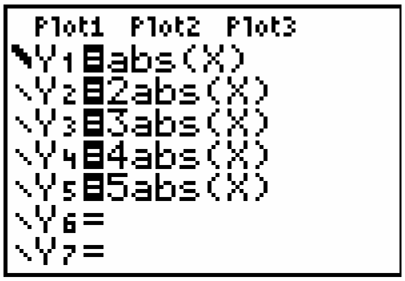
Elaborate

The Elaborate portion of the lesson provides the student with an opportunity to investigate concretely and apply the concept of transformations on the absolute value parent function.

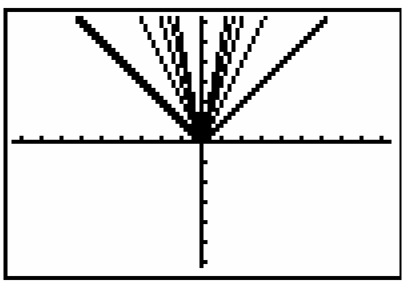
At the end of the Elaborate phase students should be able to describe the graphs of transformations to the absolute value function.

Investigating Absolute Value Functions

1. Enter the following absolute value functions into the graphing calculator.

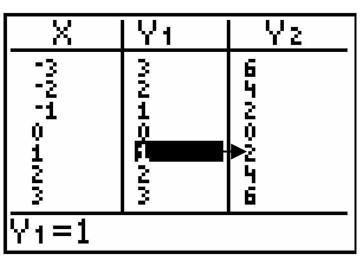


a. Sketch all five graphs on the grid below.



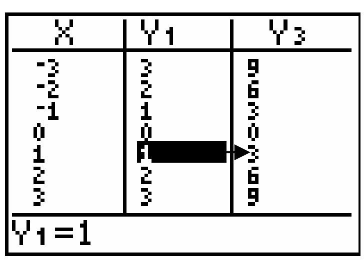
b. What do you notice about the graph as the multiplier increases?
The graph gets steeper or has a vertical stretch.

c. What do you notice about the values in the table feature of the graphing calculator?



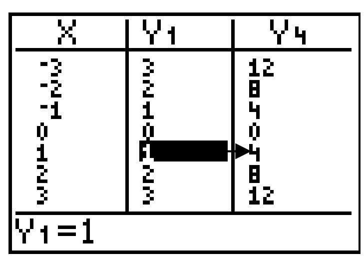
X	Y_1	Y_2
-3	3	6
-2	2	4
-1	1	2
0	0	0
1	1	2
2	2	4
3	3	6

$Y_1=1$



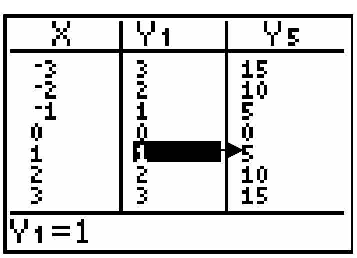
X	Y_1	Y_3
-3	3	9
-2	2	6
-1	1	3
0	0	0
1	1	3
2	2	6
3	3	9

$Y_1=1$



X	Y_1	Y_4
-3	3	12
-2	2	8
-1	1	4
0	0	0
1	1	4
2	2	8
3	3	12

$Y_1=1$

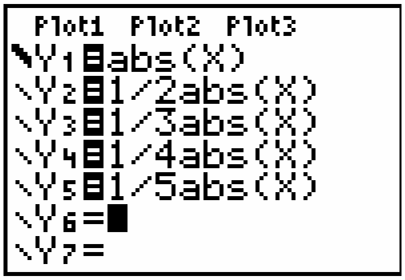


X	Y_1	Y_5
-3	3	15
-2	2	10
-1	1	5
0	0	0
1	1	5
2	2	10
3	3	15

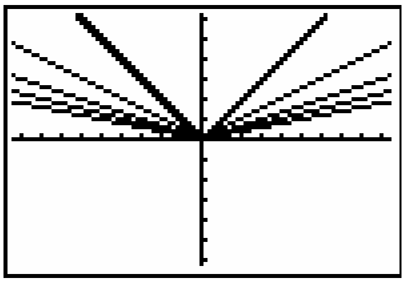
$Y_1=1$

The y-values are increased by the same scale factor as the multiplier of the parent function.

2. Enter the following absolute value functions into the graphing calculator.



a. Sketch all five graphs on the grid below.



b. What happens to each graph as the multiplier decreases, but remains positive?
The graph gets flatter or has a vertical compression.

c. Examine the tables of the five functions. What do you notice?

X	Y_1	Y_2
-3	3	1.5
-2	2	1
-1	1	.5
0	0	0
1	1	.5
2	2	1
3	3	1.5

$Y_1=1$

X	Y_1	Y_3
-3	3	1
-2	2	.66667
-1	1	.33333
0	0	0
1	1	.33333
2	2	.66667
3	3	1

$Y_1=1$

X	Y_1	Y_4
-3	3	.75
-2	2	.5
-1	1	.25
0	0	0
1	1	.25
2	2	.5
3	3	.75

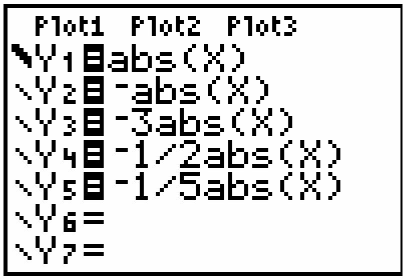
$Y_1=1$

X	Y_1	Y_5
-3	3	.6
-2	2	.4
-1	1	.2
0	0	0
1	1	.2
2	2	.4
3	3	.6

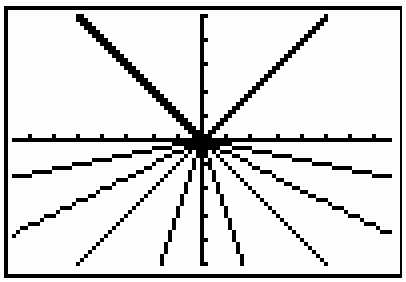
$Y_1=1$

The y-values are decreased by the same scale factor as the multiplier of the parent function.

3. Enter the following absolute value functions in the graphing calculator.



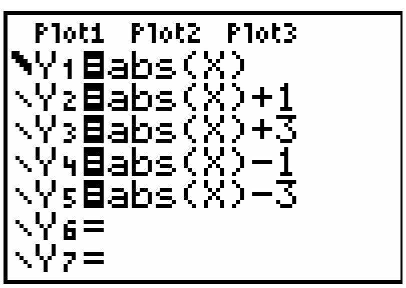
a. Sketch all five graphs on the grid below.



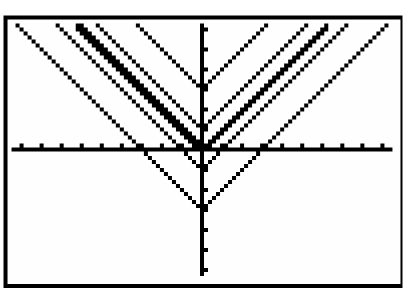
b. How does the negative multiplier appear to affect the parent function $y = |x|$?

The graphs are reflected across the x-axis and the multiplier continues to dilate the graph.

4. Enter the following absolute value functions in the graphing calculator.

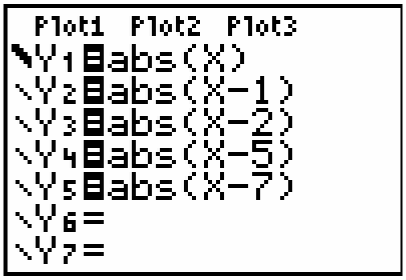


a. Sketch all five graphs on the grid below.

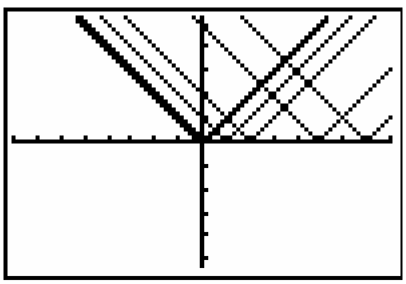


b. What do you notice about these functions?
They have been shifted vertically up or down. The amount of the shift depends on the number added to or subtracted from the absolute value function.

5. Enter the following absolute value functions in the graphing calculator.

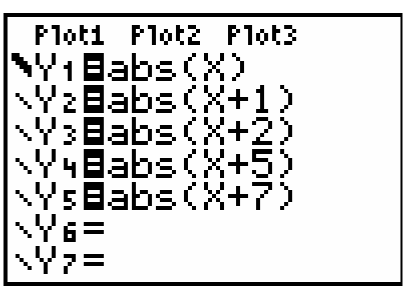


a. Sketch all five graphs on the grid below.

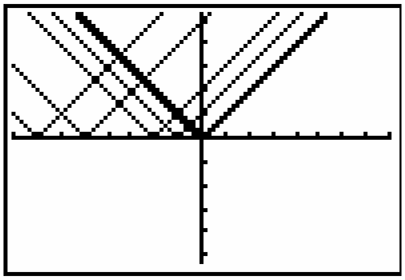


c. What do you notice about these functions?
They have been shifted to the right. The amount of the shift depends on the number inside the parentheses.

6. Enter the following absolute value functions in the graphing calculator.

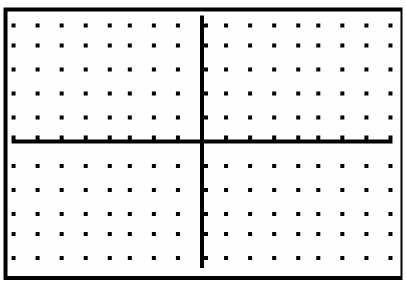


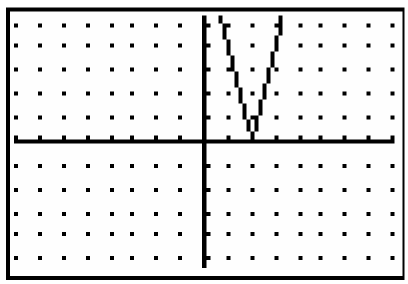
a. **Sketch all five graphs on the grid below.**



b. **What happens to each graph as the number being added to x increases?**
The graph is shifted further to the left.

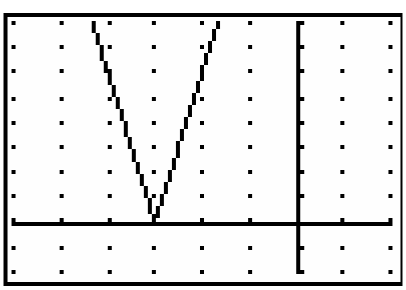
7. **Sketch a prediction of the graph of $y = 4|x - 2|$, and then check your prediction with the graphing calculator.**

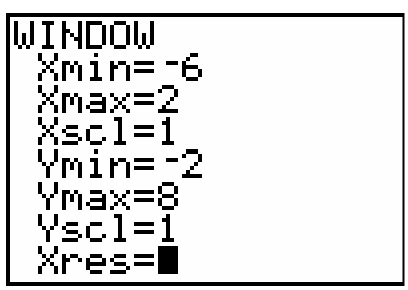




a. **How does your prediction compare with the actual graph of $y = 4|x - 2|$?**
Responses may vary.

b. **Look at the graph of the absolute value function shown below. Based on the observations you have made in this activity, predict the equation of the graph.**





Responses may vary. A sample prediction is $y = 7|x + 3|$.

c. **Graph the equation of the line you predicted, using the window. How does your prediction compare to the actual graph?**
Responses may vary.

8. Verbally describe the graphs of the functions below as compared to the graph of the parent function $y = |x|$.

a) $y = 4|x|$

vertical stretch of 4

b) $y = \frac{1}{3}|x|$

vertical compression of $\frac{1}{3}$

c) $y = 7|x-8|$

vertical stretch of 7 and
a horizontal shift 8 units to the right

d) $y = |x| + 9$

vertical shift of 9 units up

e) $y = |x+9|$

horizontal shift 9 units to the left

f) $y = -2|x| - 3$

vertical stretch of -2 units (reflects down) and a vertical shift down of 3 units

9. Using what you know about the transformations you just made and the transformations made to a quadratic function, write a general form for transformations to the absolute value parent function.

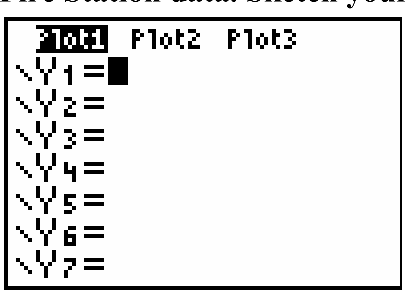
$y = a|x-h| + k$

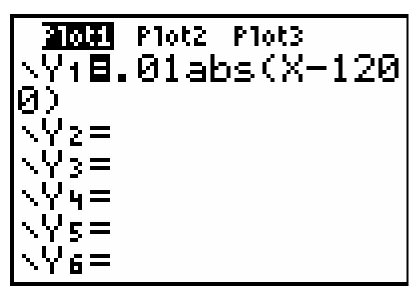
Questions 10-12 refer to The Fire Station Problem.

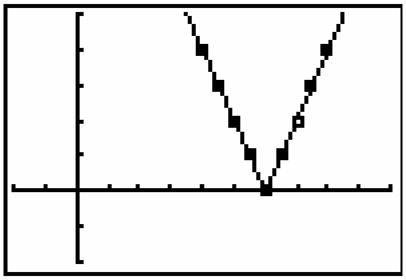
10. Think back to The Fire Station Problem and the equations we wrote to represent the data. How can we replace the two linear functions that model our data with one absolute value function?

We can take our original functions, y = 0.01(x – 1200) and y = –0.01(x – 1200), rewrite the second function to be y = 0.01[–(x – 1200)], and combine the first and last functions into one absolute value function $y = 0.01|x-1200|$ *to describe the entire set of data.*

11. Verify your conjecture by graphing the absolute value function over a scatterplot of the Fire Station data. Sketch your graph.

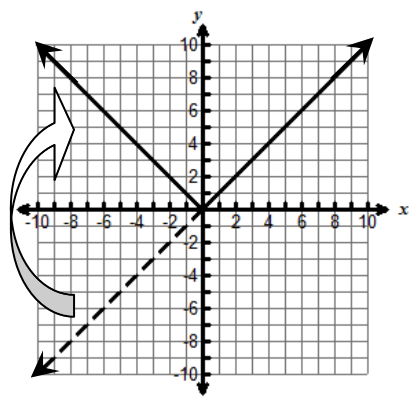






12. How does this function remedy domain restrictions we encountered by using two linear functions?

The absolute value function combines the "best attributes" of both linear functions by reflecting a portion of the linear function with a positive slope. The portion of the line to the left of the intersection point (h, k) is reflected vertically across the line y = k. The resulting shape is the v-shape of the absolute value parent function.



Evaluate

The Evaluate portion of the lesson provides the student with an opportunity to demonstrate his or her understanding of the TEKS addressed in this lesson.

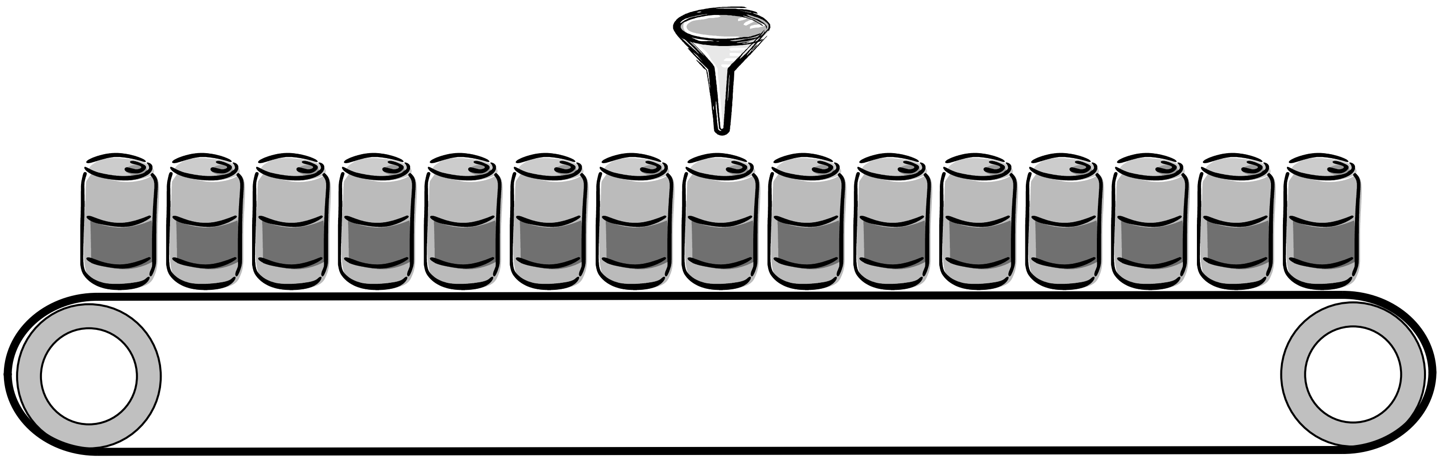
Materials Needed

- Student copies of Soda Can Performance Assessment
- Graphing calculators
- Chart paper
- markers

Duration: 30 – 45 minutes

Provide each student with a copy of the Soda Can Performance Assessment and a graphing calculator. Give students 15 – 20 minutes to work on the performance assessment independently. After the time limit is up, students work in groups of 4 and come to a consensus about the solutions. Each group must present a common solution on chart paper or individual students may turn in solutions.

Soda Cans
Performance Assessment



Soda cans move along a conveyor belt at a constant rate. When each can reaches the center of the belt, it is filled with soda. Each can has a diameter of 4 inches at the base. The first can on the left represents can # 1 and the can at the right represents can # 15. Relate the can number, *x*, and its distance from the filling spout, *y*, three different ways.

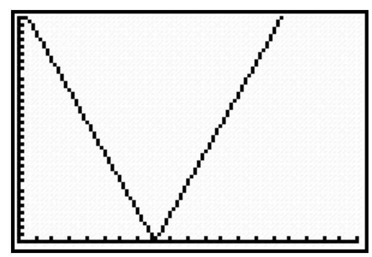
Table

Can # (*x*)	**Distance from filling spout** (*y*)
1	*28*
2	*24*
3	*20*
4	*16*
5	*12*
6	*8*
7	*4*
8	*0*
9	*4*
10	*8*
11	*12*
12	*16*
13	*20*
14	*24*
15	*28*

Function

$y = 4|x - 8|$

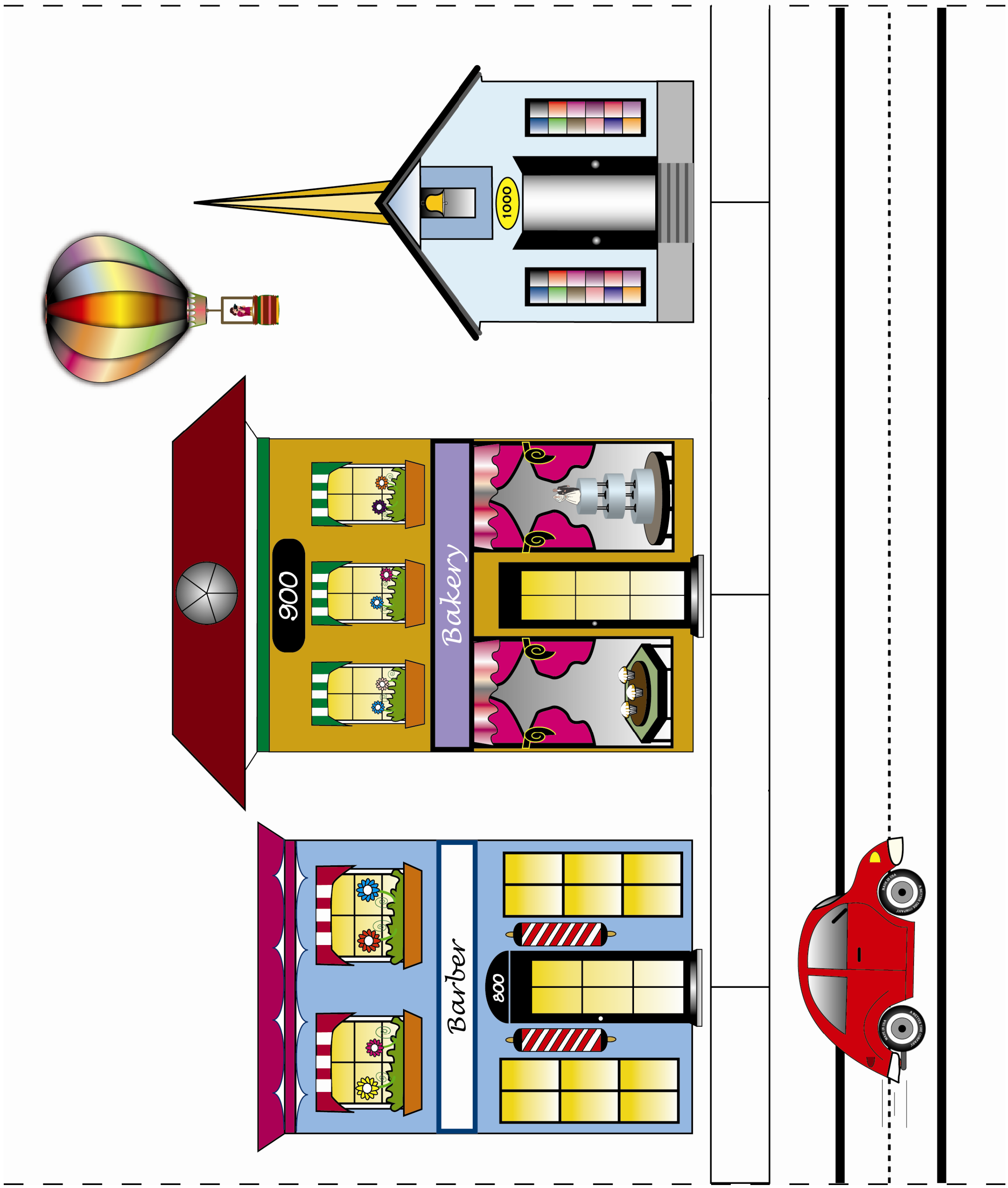
Graph



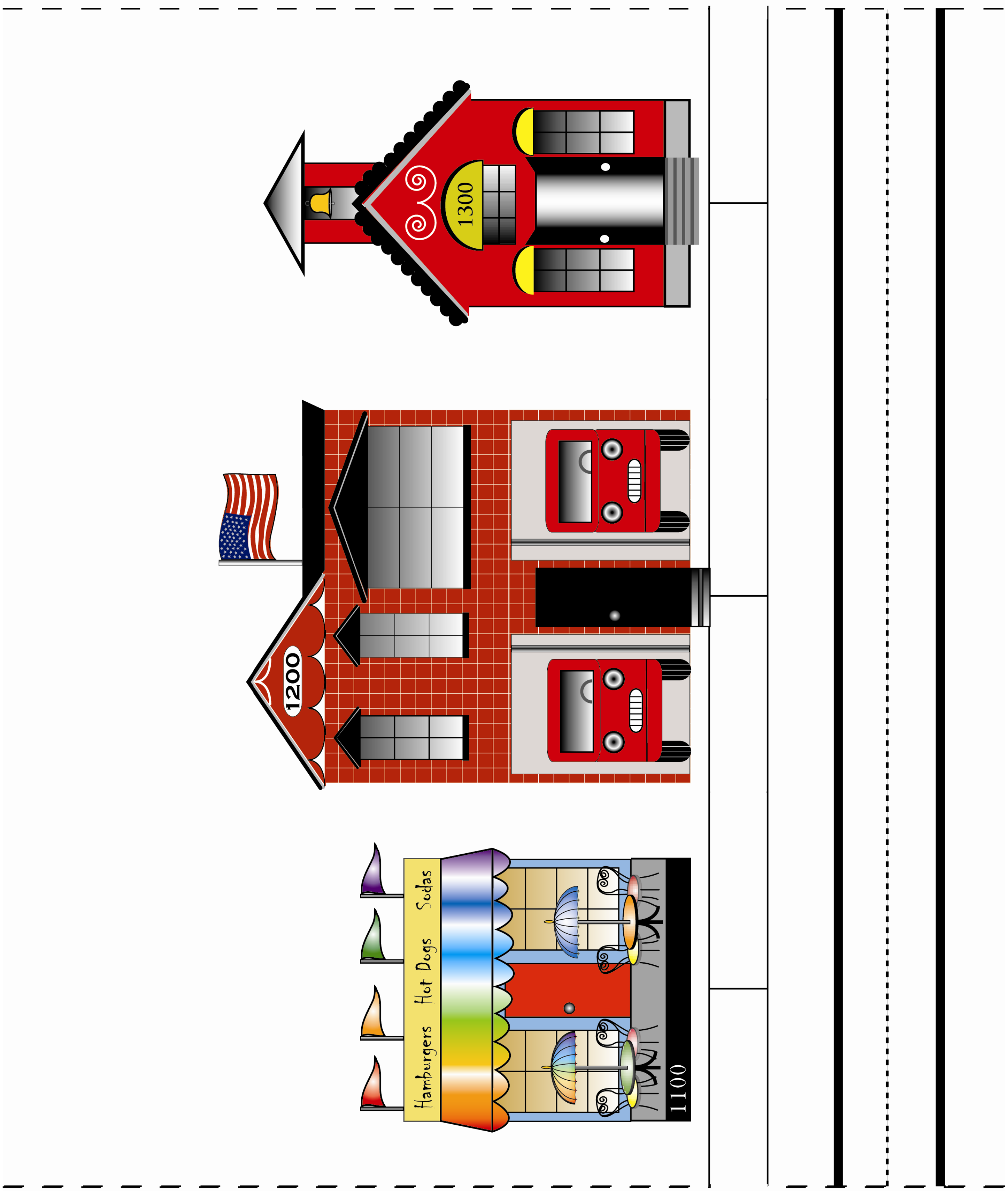
How would the graph and function change if the cans were 6 inches wide at the base and the conveyor belt held 21 bottles? $y = 6|x - 11|$

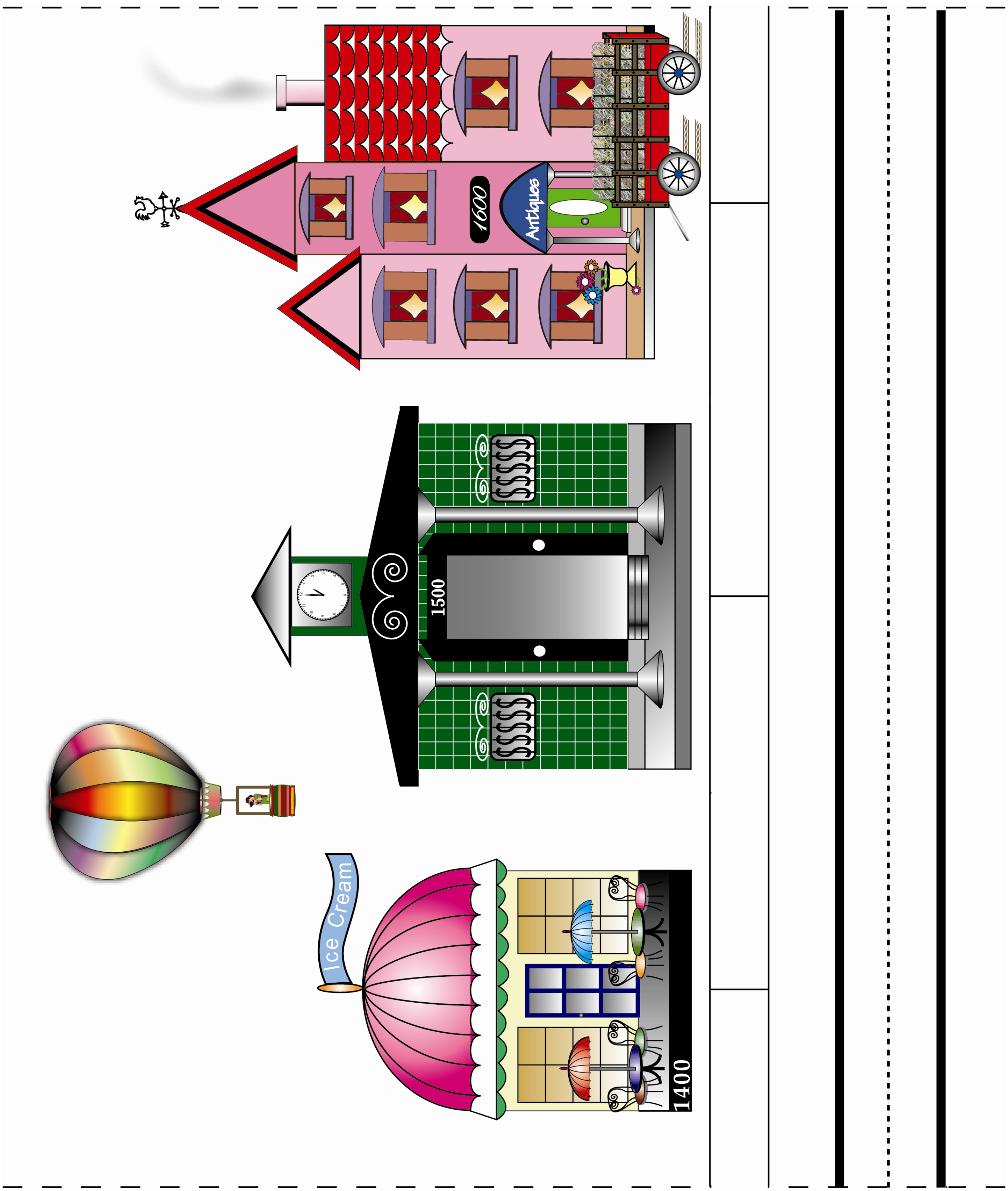
Activity Page for The Fire Station Problem

Cut along dotted line



Cut along dotted line




1600
Antiques
1500
Ice Cream
1400

The Fire Station Problem

A fire station is located on Main Street and has buildings at every block to the right and to the left. You will investigate the relationship between the address number on a building and its distance in blocks from the fire station.

1. Complete the table below that relates the address of a building (x) with its distance in blocks from the fire station (y).

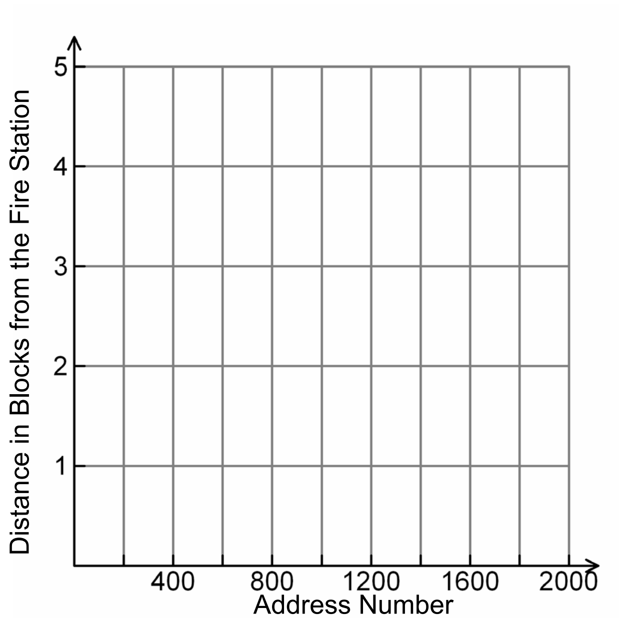
Address Number (x)	**Distance in Blocks from the Fire Station (y)**
800	
900	
1000	
1100	
1200	
1300	
1400	
1500	
1600	

2. Which building is 2 blocks away from the fire station? Explain your answer.

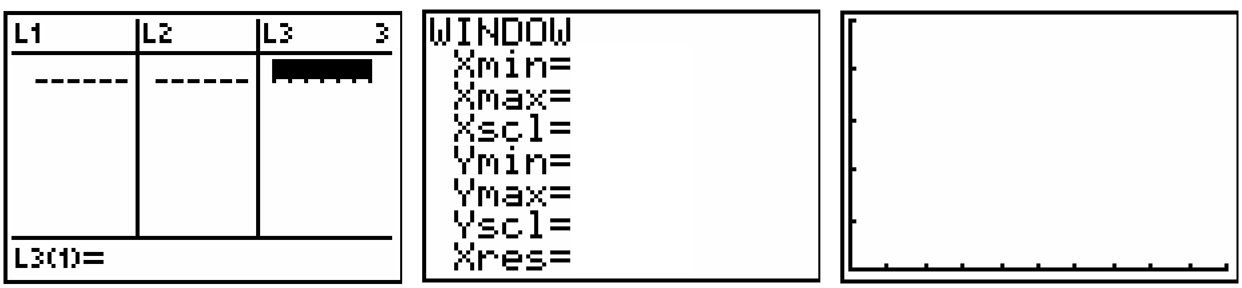
3. If we send someone to the building that is 2 blocks away from the fire station, how will she know that she has arrived at the correct place?

4. How do we describe two numbers that represent the same distance from a location?

5. Draw a scatterplot that represents the data in the table.



6. Make a scatterplot of your data using your graphing calculator. Describe your viewing window.

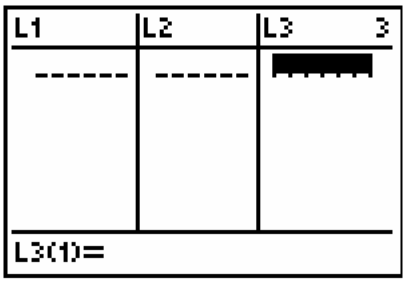


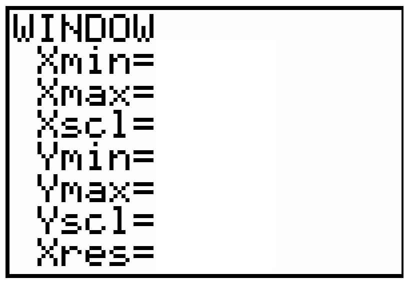
7. What function or functions might you use to describe the scatterplot?

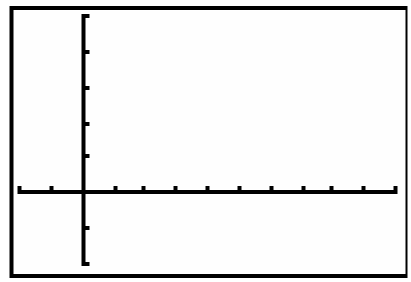
8. Find two linear functions that pass through the data points. What process did you use to find the equations of the lines?

9. Graph the equations on your calculator. How are the equations similar? How are they different?

10. If necessary adjust the window to clearly see the intersection of the two lines. What does the intersection of these two lines represent?







11. Where do the equations fit the graph of the data points? Where do the equations not fit the graph of the data points?

12. How well do the linear functions model the data points?

Part 2: The Fire Station Problem

1. What is the domain and range of each of the linear functions that model the fire station problem?

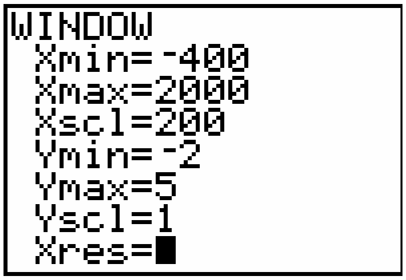
2. How do these domains and ranges compare to those of the data set?

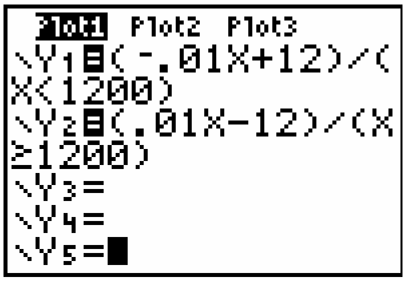
3. How do the equations compare?

4. At what point do the graphs of the equations intersect?

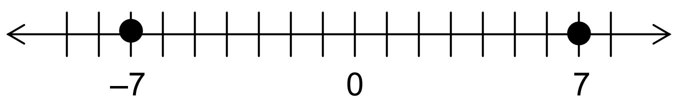
5. Which parts of the graphs of the lines model our data set? Which parts do not? Why?

6. What happens when you graph the linear functions over the given domain? Sketch and describe your resulting graph. Why do you think this happens?





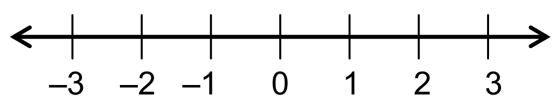
Numerically, when we consider the distance of a number from 0 on the number line, we call that distance the *absolute value* of the number.



For example, $|-7| = 7$, since –7 is 7 units to the left of 0. Similarly, $|7| = 7$, since 7 is 7 units to the right of 0.

7. The *x*-coordinates below represent locations on a number line. The *y*-coordinates represent the distance that location is from 0. For the given *x*-values, use the number line to find the corresponding *y*-values.

x	***y***
–3	
–2	
–1	
0	
1	
2	
3	



8. Make a scatterplot of *y* versus *x* on your graphing calculator. Describe your window. Sketch your scatterplot.

9. What is the shape of your scatterplot?

10. Find two linear functions that model this situation. Graph these functions.

11. Restrict the domain so that the model is an even better fit. Graph the restricted functions.

12. Now try $y_3 = |x|$ using the trace bubble. How does this function compare to the two linear functions?

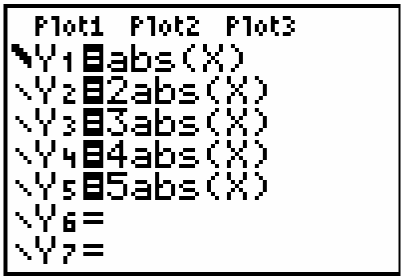
13. This is a new parent function to add to your list of parent functions. With which other parent functions are you familiar?

14. What is the parent function for absolute value?

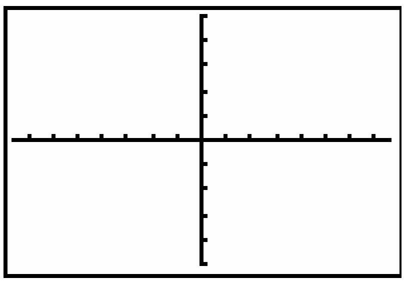
15. What are some characteristics of absolute value functions?

Investigating Absolute Value Functions

1. Enter the following absolute value functions into the graphing calculator.



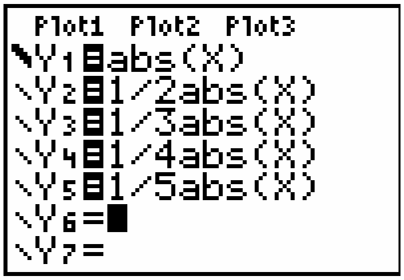
   a. Sketch all five graphs on the grid below.



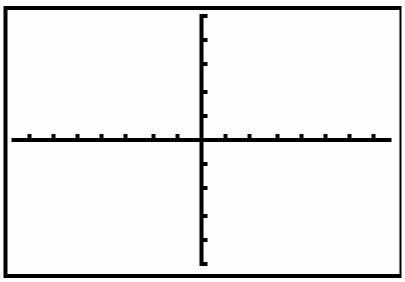
   b. What do you notice about the graph as the multiplier increases?

   c. What do you notice about the values in the table feature of the graphing calculator?

2. Enter the following absolute value functions into the graphing calculator.



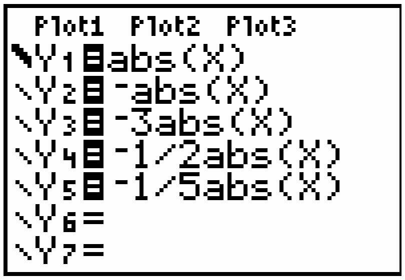
   a. Sketch all five graphs on the grid below.



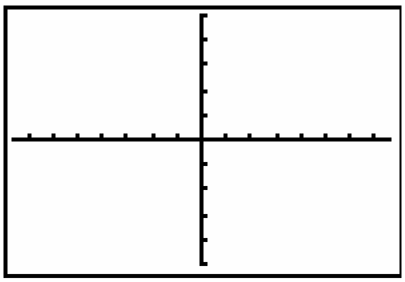
   b. What happens to each graph as the multiplier decreases, but remains positive?

   c. Examine the tables of the five functions. What do you notice?

3. Enter the following absolute value functions in the graphing calculator.

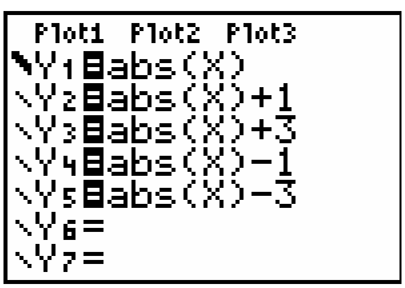


   a. Sketch all five graphs on the grid below.

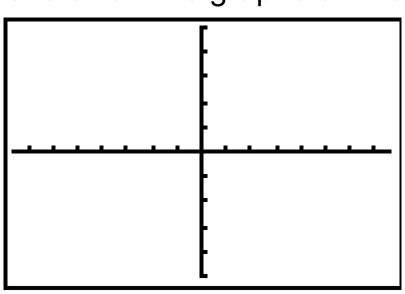


   b. How does the negative multiplier appear to affect the parent function $y = |x|$?

4. Enter the following absolute value functions in the graphing calculator.

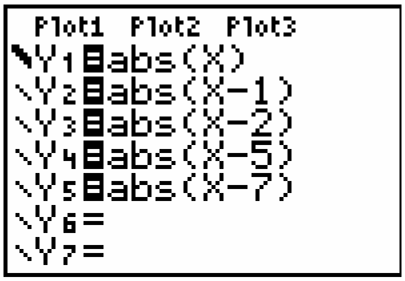


   a. Sketch all five graphs on the grid below.

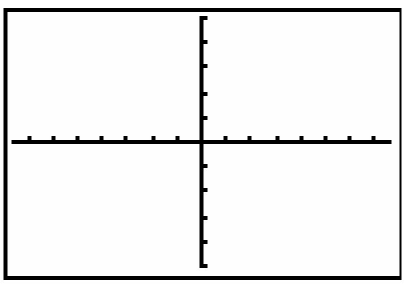


b. What do you notice about these functions?

5. Enter the following absolute value functions in the graphing calculator.

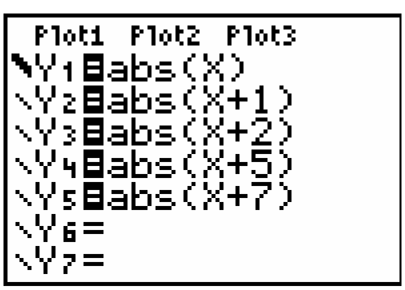


a. Sketch all five graphs on the grid below.

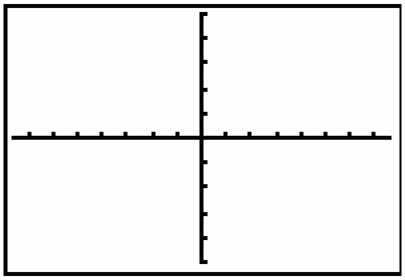


b. What do you notice about these functions?

6. Enter the following absolute value functions in the graphing calculator.

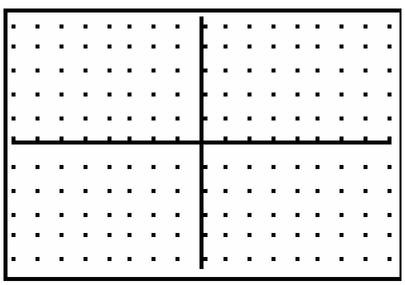


a. Sketch all five graphs on the grid below.



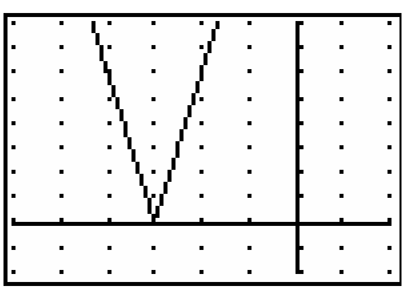
b. What happens to each graph as the number being added to x increases?

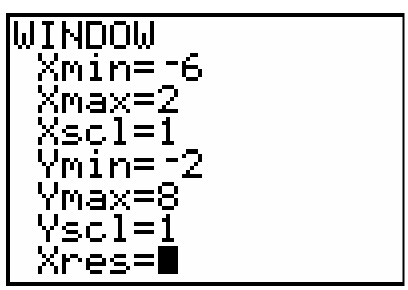
7. Sketch a prediction of the graph of $y = 4|x - 2|$, and then check your prediction with the graphing calculator.



a. How does your prediction compare with the actual graph of $y = 4|x - 2|$?

b. Look at the graph of the absolute value function shown below. Based on the observations you have made in this activity, predict the equation of the graph.





c. Graph the equation of the line you predicted, using the window. How does your prediction compare to the actual graph?

8. Verbally describe the graphs of the functions below as compared to the graph of the parent function $y = |x|$.

a) $y = 4|x|$

b) $y = \frac{1}{3}|x|$

c) $y = 7|x - 8|$

d) $y = |x + 9|$

e) $y = |x + 9|$

f) $y = -2|x| - 3$

9. Using what you know about the transformations you just made and the transformations made to a quadratic function, write a general form for transformations to the absolute value parent function.

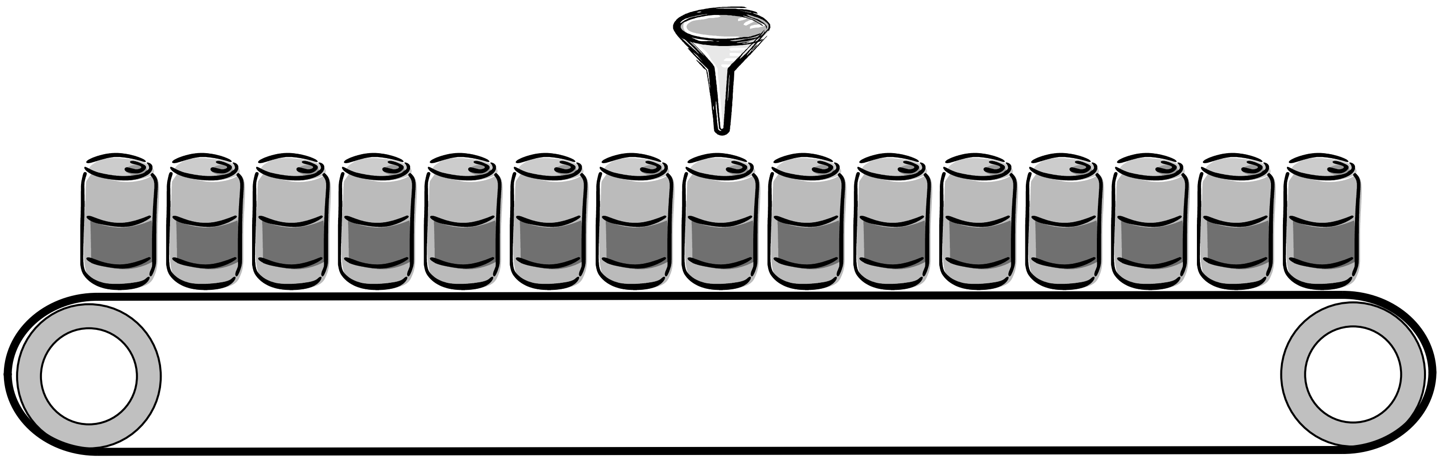
Questions 10-12 refer to The Fire Station Problem.

10. Think back to The Fire Station Problem and the equations we wrote to represent the data. How can we replace the two linear functions that model our data with one absolute value function?

11. Verify your conjecture by graphing the absolute value function over a scatterplot of the Fire Station data. Sketch your graph.

12. How does this function remedy domain restrictions we encountered by using two linear functions?

Soda Cans
Performance Assessment



Soda cans move along a conveyor belt at a constant rate. When each can reaches the center of the belt, it is filled with soda. Each can has a diameter of 4 inches at the base. The first can on the left represents can # 1 and the can at the right represents can # 15. Relate the can number, *x,* and its distance from the filling spout, *y*, three different ways.

Table

Can Number (*x*)	Distance from filling spout (*y*)
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

Function

Graph

How would the graph and function change if the cans were 6 inches wide at the base and the conveyor belt held 21 bottles?

Student Lesson: Rational Functions

Purpose:
At the end of this lesson students should be able to identify the reciprocal function and any transformations to the function. This is the initial introduction to the concept of a rational function. After this lesson, students would continue their study of rational functions by graphing rational functions, solving rational equations, and solving problem situations modeled by rational functions.

TEKS:
a5 Tools for algebraic thinking. Techniques for working with functions and equations are essential in understanding underlying relationships. Students use a variety of representations (concrete, pictorial, numerical, symbolic, graphical, and verbal), tools, and technology (including, but not limited to, calculators with graphing capabilities, data collection devices, and computers) to model mathematical situations to solve meaningful problems.

a6 Underlying mathematical processes. Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics. Students also use multiple representations, technology, applications and modeling, and numerical fluency in problem-solving contexts.

2A.10 **Rational Functions.** The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

2A.10A The student is expected to use quotients of polynomials to describe the graphs of rational functions, predict the effects of parameter changes, describe limitations on the domains and ranges, and examine asymptotic behavior.

2A.10B The student is expected to analyze various representations of rational functions with respect to problem situations.

2A.10C The student is expected to determine the reasonable domain and range values of rational functions, as well as interpret and determine the reasonableness of solutions to rational equations and inequalities.

2A.10G The student is expected to use functions to model and make predictions in problem situations involving direct and inverse variation.

TAKS™ Objectives Supported:
While the Algebra II TEKS are not tested on TAKS, the concepts addressed in this lesson reinforce the understanding of the following objectives.

- Objective 1: Functional Relationships
- Objective 2: Properties and Attributes of Functions
- Objective 10: Mathematical Processes and Mathematical Tools

Materials

Prepare in Advance:

- Copies of student activity pages, **How are They Alike? How are They Different?, Fallingwater, Making a Spaghetti Cantilever**, **Understanding the Reciprocal Function**, and **Transformations on the Reciprocal Function**
- Copies of the **Deflection Measurement Grid** (optional)
- Transparencies: **Class Data for Spaghetti Cantilevers**

For each student group of 3 - 4 students:

- Film canister (35mm) without lid
- 15 pennies (preferably all minted within the last 10 years)
- 1 paper clip (optional)
- 50 pieces of spaghetti from a newly opened package (all groups should use same type and brand of spaghetti – linguine or fettuccini may be used in place of spaghetti)
- Masking tape and string
- Scissors
- Graphing calculator
- Meter stick and ruler
- **Spaghetti Cantilever Activity Recording Sheet**

For each student:

- Graphing calculator
- **Making a Spaghetti Cantilever** activity sheet
- **Understanding the Reciprocal Function** activity sheet
- **Transformations on the Reciprocal Function** activity sheet
- **Let's Be Rational!** Assessment card match and answer sheet
- 2 or 3 pencils of various colors

Engage

The Engage portion of the lesson is designed to generate student interest in the mathematics involved in the cantilever, providing a real-life example of inverse functional relationships.

Pass out the activity page, **How Are They Alike? How Are They Different?** Ask students to describe any similarities and differences they notice. Hopefully, students will notice that some of the objects are supported at only one end while others are supported at two or more places.

Pass out the activity page, **Fallingwater**. Show the photograph of the Frank Lloyd Wright designed home, Fallingwater, and ask students to share ideas about the design of the home. You would like for students to notice that the porches are supported at only one end. Most students will not be familiar with the vocabulary "cantilever". Before you use the word, you might want to play a game of Hangman with the students with cantilever in order for them to answer the question, "Can you think of a word that describes an object that is supported at only one end?"

Facilitation Questions

- **Have you ever seen a house similar to this one in appearance?**
  Responses may vary; however, it is unlikely that students have seen many homes like this one.

- **What do you notice about the house?**
  There are many characteristics students may notice, such as the waterfall directly under the house, lots of windows, or porches that seem to extend with no support.

- **What do you notice about the porches?**
  One thing that you want the students to notice is the porches. Some of them are horizontally extended with little visible support.

- **Can you think of any other examples of cantilevers?**
  Responses may vary, but hopefully students may come up with a crane, bridge, telephone or electrical poles, or some chairs or desks have a cantilever design.

Explore

The Explore portion of the lesson provides the student with an opportunity to be actively involved in the exploration of the mathematical concepts addressed. Students will participate in an investigation of data that they collect and interpret the data in the context of a cantilever's thickness and deflection. This part of the lesson is designed for groups of three to four students.

1. Distribute the **Making a Spaghetti Cantilever** instruction sheet. Allow students a few minutes to silently read through the introduction and instructions. Ask for student interpretations of the investigation and answer any questions about the set-up for the investigation. You may want to prepare the film canister baskets ahead of time to save time in class.

2. Distribute the **Spaghetti Cantilever Activity Recording Sheet**. Have students follow the instructions to collect data and record the deflection measurements on their group data table. An optional way to measure deflection is included. If you would like, you can have students measure deflection with the Deflection measurement grid.

3. Have each group plot their results on a sheet of chart paper.

4. When each group has finished collecting the data and making the chart, have the Data manager record their measurements on the **Class Data for Spaghetti Cantilevers** transparency.

5. Student groups should work together to complete the questions on the recording sheet. Monitor students' progress as they work on the recording sheets, helping as needed so that all groups can finish within a reasonable amount of time. Debrief the student responses after all groups have finished the activity.

Spaghetti Cantilever Activity Recording Sheet

Answer Key

1. **Fill in the table below with the data you collected.**
   Sample responses:

Number of Pieces of Spaghetti in the Bundle (x)	Amount of Deflection in the Spaghetti (cm) (y)	Product of x and y ($x \cdot y$)
1	*8.0*	*8.0*
2	*6.8*	*13.6*
3	*4.9*	*14.7*
4	*3.6*	*14.4*
5	*3.2*	*16*
6	*3.0*	*18.0*
7	*2.8*	*19.6*
8	*2.5*	*20.0*

2. What are the two quantities which vary in this investigation?

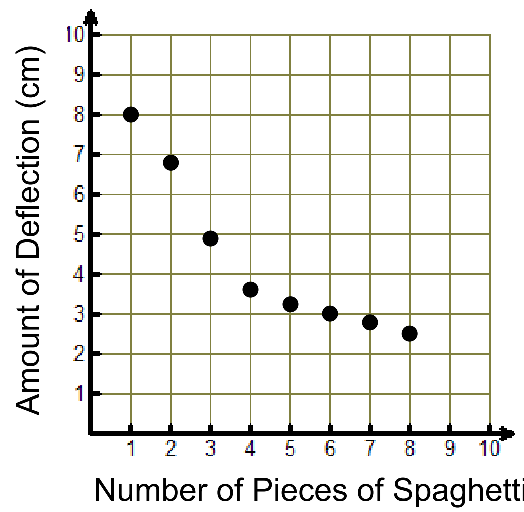
***x* represents** *the number of pieces of spaghetti in each bundle*.

***y* represents** *the number of centimeters which the spaghetti bundle was deflected*.

3. Write a dependency statement relating the two variables. Use the quantities described above, not just "*x*" and "*y*".
The amount of deflection depends on the number of pieces of spaghetti in a bundle.

4. What are a reasonable domain and range for the set of data?
The domain is 1, 2, 3, 4, 5, 6, 7, and 8.
The range is between 2.5 and 8.

5. Enter the data you collected into your graphing calculator and sketch the resulting scatterplot on the grid provided.



6. What happens to the amount of deflection as the number of pieces of spaghetti increases?
As the number of pieces of spaghetti increases then deflection decreases.

7. Is this data set discrete or continuous? Why?
This is discrete data because it would not make sense to have fractional pieces of spaghetti in a bundle – at least not every fraction between 1 and 8.

8. Does the set of data represent a function? Why?
Yes, the set of data is a function. For every x value, there is only one y value.

9. What kind of function would best fit your data and scatterplot? Why?
This looks like a rational (or inverse) function. The form is $y = \frac{a}{x}$. *The graph is decreasing like a rational function and seems to level off – possibly having a vertical and horizontal asymptote.*

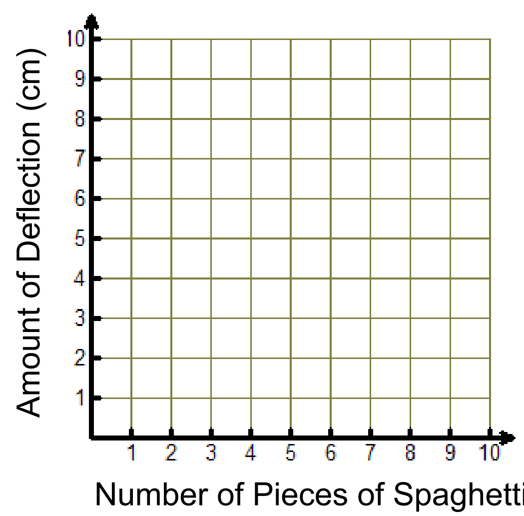
10. Use what you know about inverse relationships to write a function rule for the spaghetti cantilever data.
If we average the xy products from the data we get about 15.5 so the equation can be $y = \frac{15.5}{x}$.

11. To get a better model add your set of data to the data of the entire class. Each group should send its Data manager to the overhead to fill in the data collected for your group. Record the additional data in the table below. Find the average deflection for each bundle of spaghetti for the entire class.
Answers will vary but hopefully the class data will be a better fit for an inverse function.

Number of Pieces of Spaghetti in the Bundle	Amount of Deflection for Each Team												Average
	A	B	C	D	E	F	G	H	I	J	K	L	
1													
2													
3													
4													
5													
6													
7													
8													

12. Using data from the entire class, create a scatterplot of deflection vs. number of pieces of spaghetti. How does this new scatterplot compare to the plot you made with just your group's data?
Answer depends on class data.

13. Write a new function rule for the class data.
Answer depends on class data.



Explain

The Explain portion of the lesson is facilitated by the teacher to allow the students to formalize their understanding of the TEKS addressed in the lesson. Students may work in small groups or pairs with the teacher debriefing when all or most students have had time to complete the activity sheet.

Distribute the **Understanding the Reciprocal Function** activity sheet. Lead students through the questions on the activity sheet but allow time for them to process and answer each question before eliciting responses.

Understanding the Reciprocal Function

Answer Key

1. **What is the reciprocal of the linear parent function, $f(x) = x$?**

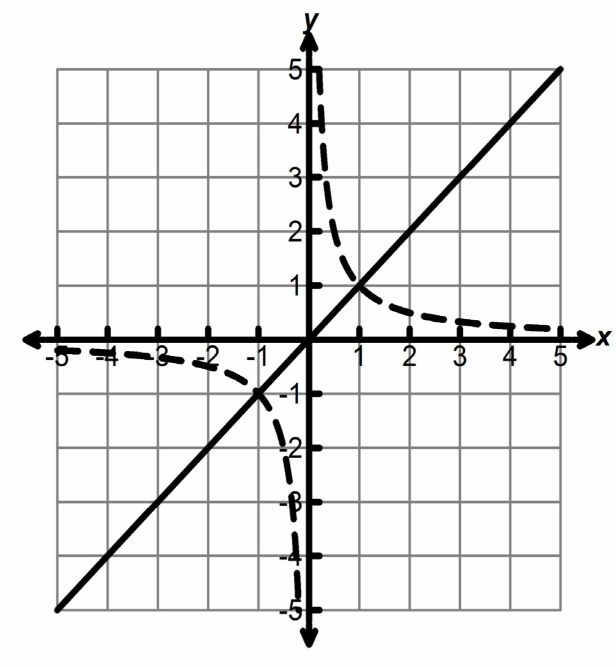
   Hint: The reciprocal of 2 is $\frac{1}{2}$ *so think of the reciprocal of x.*

   The reciprocal of f(x) = x is $f(x) = \frac{1}{x}$.

2. **Let's investigate some of the attributes of the function $f(x) = x$ and its reciprocal, $g(x) = \frac{1}{x}$. Complete the tables below for each function. Draw a sketch of the graphs of the two functions on the same set of axes.**

$f(x) = x$	
x	***y***
–3	*–3*
–2	*–2*
–1	*–1*
–0.5	*–0.5*
–0.1	*–0.1*
0	*0*
0.1	*0.1*
0.5	*0.5*
1	*1*
2	*2*
3	*3*

$g(x) = \frac{1}{x}$	
x	***y***
–3	*–1/3*
–2	*–1/2*
–1	*–1*
–0.5	*–2*
–0.1	*–10*
0	*Und*
0.1	*10*
0.5	*2*
1	*1*
2	*1/2*
3	*1/3*



3. Where do the functions $f(x) = x$ and its reciprocal function $g(x) = \frac{1}{x}$ intersect? Why?

The graphs intersect at (1, 1) and (–1, –1) because for both –1 and 1, they are equal to themselves and they are their own reciprocals.

4. For any given *x* value, compare the $f(x)$ and $g(x)$ values. Write a verbal description of how these function values are related.

All of the function values for g(x) are the reciprocals of the function values for f(x).

5. Using your graphing calculator (if necessary), fill in the tables below. Let $f(x) = x$ be Y_1, and let $g(x) = \frac{1}{x}$ be Y_2.

$Y_1 = x$		$Y_2 = \frac{1}{x}$
$(-\infty, \infty)$	**Intervals where the function is increasing**	*none*
none	**Intervals where the function is decreasing**	$(-\infty, 0) \cup (0, \infty)$
none	**Intervals where the function is undefined**	*x = 0*
(0, 0)	**Coordinates of the *x*-intercepts (zeros)**	*none*
none	**Equations of any asymptotes**	*x = 0, y = 0*

6. What do you notice about the graphs of the linear parent function and it's reciprocal?

They seem to have opposite characteristics, such as one is increasing for all x values while the other is decreasing. One has a zero at (0, 0) while the other is asymptotic at x = 0 and y = 0.

7. How does each function behave for values of *x* that are very close to 0?

Hint: Use your graphing calculator table in Ask Mode to investigate x-values between 0 and 1. Check out the negatives values between –1 and 0 also.

For f(x) = x, the y values are very small when the x values are close to zero; x and y are equal.

For $g(x) = \frac{1}{x}$, *the y values are very large when the x values are close to zero.*

8. How does each function behave as the values of *x* get very large?
Hint: Use your graphing calculator table in Ask Mode to investigate x-values such as 10, 100, 1000, 10000. Check out negative values –10, –100, etc. as well.

For f(x) = x, the y values are very large when the x values are very large; x and y are equal.
For $g(x) = \frac{1}{x}$*, the y values are very close to zero when the x values are very large.*

9. How would you describe an asymptote to a friend?
Answers may vary. The distance between an asymptotes and a point (x, y) on a graph approaches zero as x gets very large or very small.

10. What are some other names for the parent function, $y = \frac{1}{x}$ **?**

The reciprocal function is also called the inverse function when xy = a, and a rational function because the function rule is a rational expression.

Elaborate

The Elaborate portion of the lesson provides an opportunity for the student to apply the concepts of the TEKS within a new situation. In this lesson, students will compare different rational functions, concentrating on the effects of a, h, and k, in the function $y = \frac{a}{(x-h)} + k$ *. Students may work in small groups or pairs with the teacher debriefing when all or most student have had time to complete the activity sheet.*

Distribute the **Transformations on the Reciprocal Function** activity sheet. Students should follow the directions to complete the activity sheet. Monitor student progress as students work.

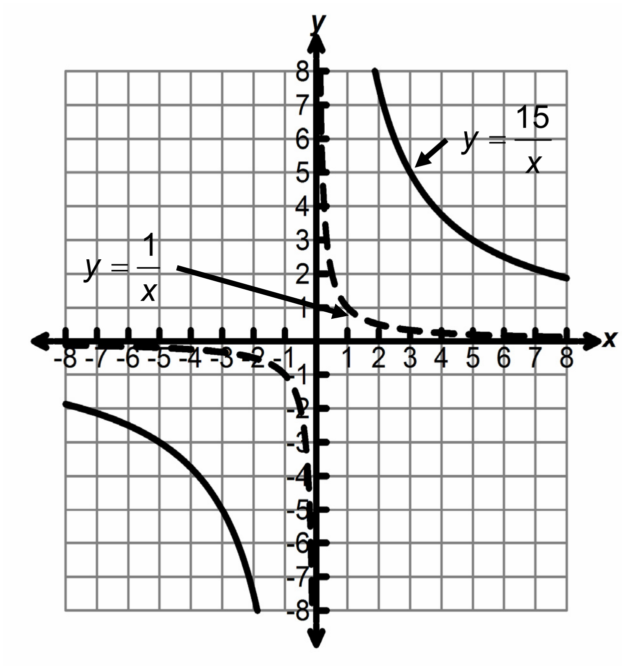
Transformations to the Reciprocal Parent Function

Answer Key

1. Use your graphing calculator to compare the two functions.

$Y_1 = \frac{1}{x} \qquad Y_2 = \frac{15}{x}$

Sketch the graphs on the coordinate axes using two differently colored pencils.



2. Complete the table below for the given values of *x*.

x	$\frac{1}{x}$	$\frac{15}{x}$
1	*1*	*15*
2	*1/2*	*7.5*
3	*1/3*	*5*
5	*1/5*	*3*

3. For each *x*-value in the table how does the value of $\frac{15}{x}$ compare with the value of $\frac{1}{x}$? Why?

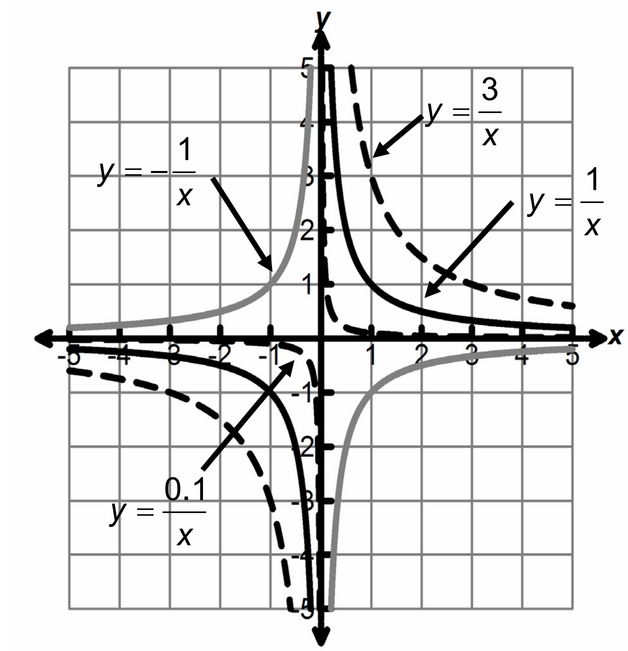
Each y value for $y = \frac{15}{x}$ *is 15 times the y value for* $y = \frac{1}{x}$ *because the* $\frac{15}{x}$ *is the same as* $15 \bullet \frac{1}{x}$.

4. The function $y = \frac{15}{x}$ is said to be a *transformation* of the parent reciprocal function $y = \frac{1}{x}$. The numerator value is 15 times greater than the numerator value in the parent function. How does this transformation affect the graph of the parent function? Why?

All of the y values are 15 times greater, so the graph is stretched vertically by a factor of 15.

5. Use your graphing calculator (if necessary) to complete the tables, explore the following functions, sketch their graphs using differently colored pencils, then answer the questions that follow.

x	$Y_1 = \frac{1}{x}$	$Y_2 = \frac{3}{x}$	$Y_3 = \frac{0.1}{x}$	$Y_1 = -\frac{1}{x}$
−4	*–1/4*	*–3/4*	*–1/40*	*1/4*
−3	*–1/3*	*–1*	*–1/30*	*1/3*
−2	*–1/2*	*–3/2*	*–1/20*	*1/2*
−1	*–1*	*–3*	*–1/10*	*1*
0	*Und*	*Und*	*Und*	*Und*
1	*1*	*3*	*1/10*	*–1*
2	*1/2*	*3/2*	*1/20*	*–1/2*
3	*1/3*	*1*	*1/30*	*–1/3*
4	*1/4*	*3/4*	*1/40*	*–1/4*



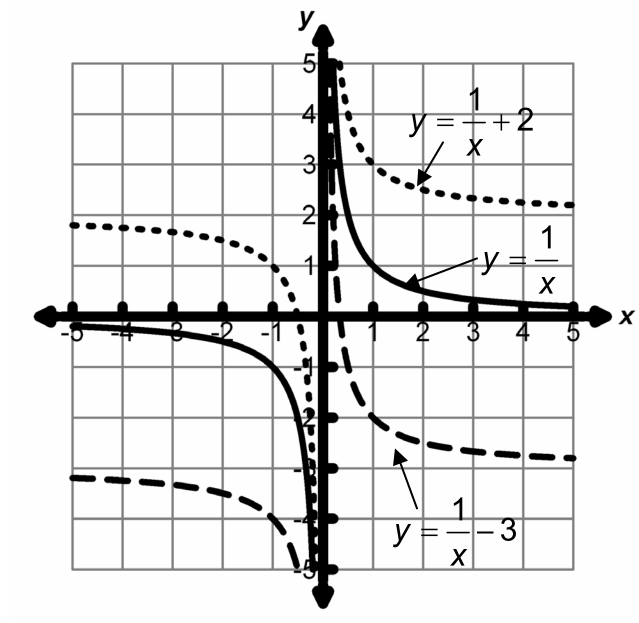
6. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when it is multiplied by a constant greater than 1?
The graph is stretched vertically but has the same asymptotes. The graph's "bend" is not as close in to the origin as the parent function.

7. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when it is multiplied by a constant between 0 and 1?
The graph is compressed vertically so that it seems to curve in closer to the origin.

8. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when it is multiplied by a negative constant?
The graph is flipped vertically, that is, it is reflected over the x-axis.

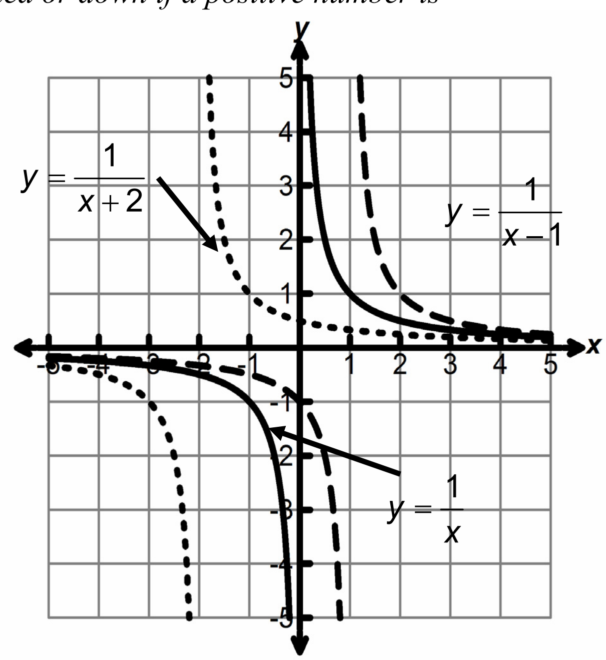
9. Use your graphing calculator (if necessary) to complete the tables, sketch the graphs of the functions from each table using differently colored pencils, then answer the questions that follow.

x	$Y_1 = \frac{1}{x}$	$Y_2 = \frac{1}{x} + 2$	$Y_3 = \frac{1}{x} - 3$
−4	*–1/4*	*1 3/4*	*–3 1/4*
−3	*–1/3*	*1 2/3*	*–3 1/3*
−2	*–1/2*	*1 1/2*	*–3 1/2*
−1	*–1*	*1*	*–4*
0	*Und*	*Und*	*Und*
1	*1*	*3*	*–2*
2	*1/2*	*2 1/2*	*–2 1/2*
3	*1/3*	*2 1/3*	*–2 2/3*
4	*1/4*	*2 1/4*	*–2 3/4*



10. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when a constant is added to or subtracted from the function rule? How can you see the movement in your table?
The graph is shifted up if a positive number is added or down if a positive number is subtracted.

x	$Y_1 = \frac{1}{x}$	$Y_2 = \frac{1}{x+2}$	$Y_3 = \frac{1}{x-1}$
−4	*–1/4*	*–1/2*	*–1/5*
−3	*–1/3*	*–1*	*–1/4*
−2	*–1/2*	*Und*	*–1/3*
−1	*–1*	*1*	*–1/2*
0	*Und*	*1/2*	*–1*
1	*1*	*1/3*	*Und*
2	*1/2*	*1/4*	*1*
3	*1/3*	*1/5*	*1/2*
4	*1/4*	*1/6*	*1/3*



11. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when a constant is added to or subtracted from the *x* variable? How can you see the movement in your table?
The graph is shifted left if a positive number is added or right if a positive number is subtracted.

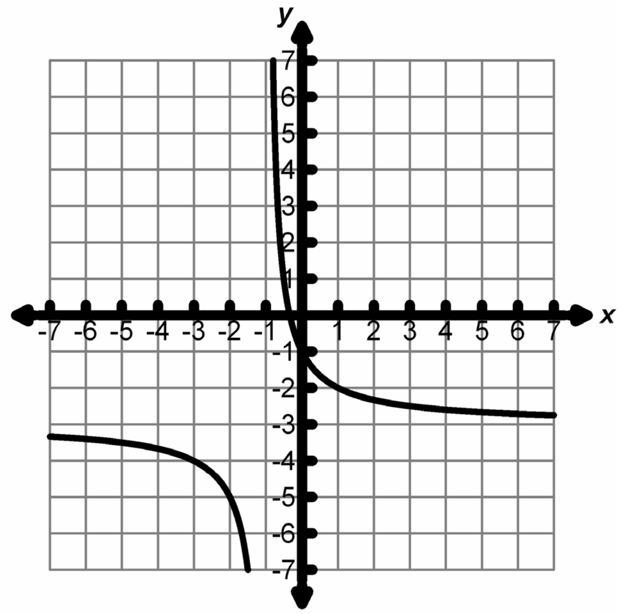
12. Do any of the changes described above affect the asymptotes of the parent reciprocal function? Explain.
Yes, the asymptotes are shifted the same distance and direction as all of the points on the graph.

13. Do any of the changes described above affect the domain and range of the reciprocal parent function? Explain.
Yes, the domain and range are shifted the same distance and direction as all of the points on the graph.

14. How are the transformations to the reciprocal function like transformations to the quadratic parent function? How are they different?
Possible responses may include: Multiplying by a constant vertically dilates (stretches or compresses) the parent function for both reciprocal and quadratic functions. Adding a constant to the function slides the function vertically for both parent functions. Adding a constant to the x term shifts the function to the left or right. The properties of the transformations are the same even though the parent functions look very different.

15. To summarize all of the transformations that you have studied here, look at the function below. Predict, describe and then sketch the effects of the transformations to the parent reciprocal function. Also describe any changes to the asymptotes, domain, and range.

$$f(x) = \frac{2}{x+1} - 3$$



Asymptotes: *vertical, x = –1*
horizontal, y = –3

Domain: $(-\infty, -1) \cup (-1, \infty)$
All reals, x ≠ –1

Range: $(-\infty, -3) \cup (-3, \infty)$
All reals, y ≠ –3

Evaluate

The Evaluate portion of the lesson provides the student with an opportunity to demonstrate his or her understanding of the TEKS addressed in the lesson.

Provide each student with a copy of the **Let's Be Rational!** assessment. Students will match a graph and table with each of four rational functions. If you wish you may cut out the tables and graphs and use the assessment as a card sort activity. There are two extra graphs and tables given. These will not be used.

Let's Be Rational!

Answer Key

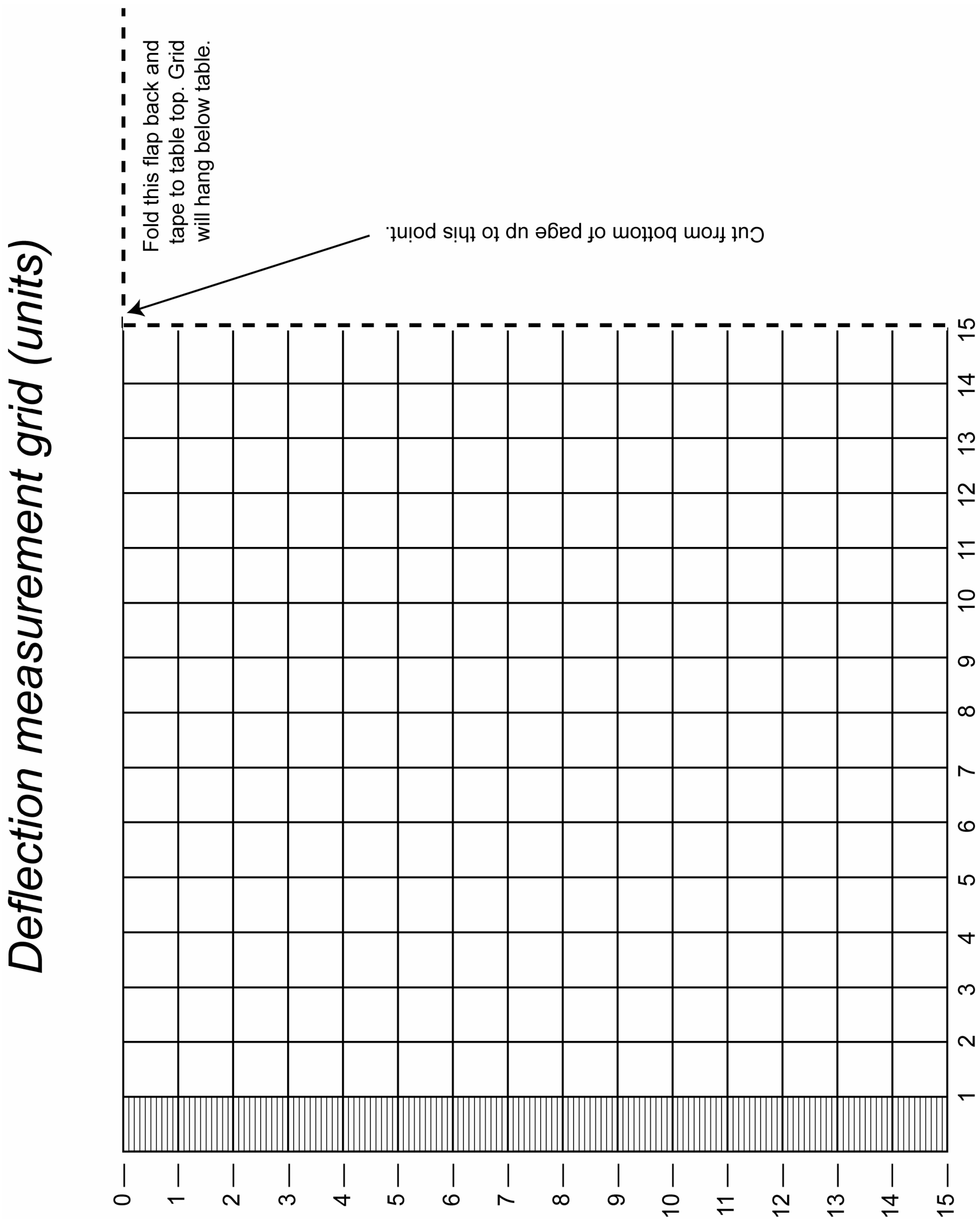
For functions 1 – 4 below, match each function to its graph and matching table of values. Write the letter of the graph and table in the grid provided. (If you want to copy the tables, graphs, and equations onto cardstock this assessment could also be done as a card match.)

	Function	Table *Write the letter*	Graph *Write the letter*
1	$y = \dfrac{1}{x+2}$	*B*	*L*
2	$y = \dfrac{1}{x} + 2$	*C*	*I*
3	$y = \dfrac{1}{x} - 2$	*F*	*J*
4	$y = -\dfrac{2}{x}$	*A*	*K*

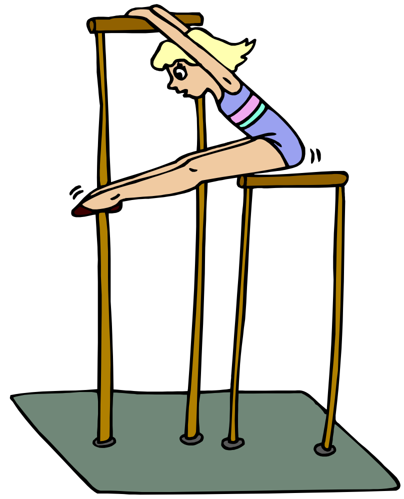
Transparency

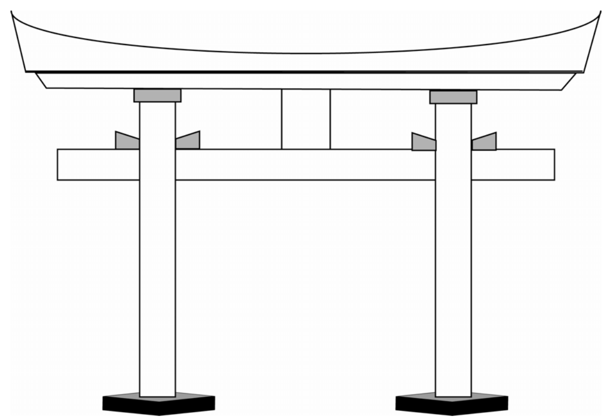
Class Data for Spaghetti Cantilevers

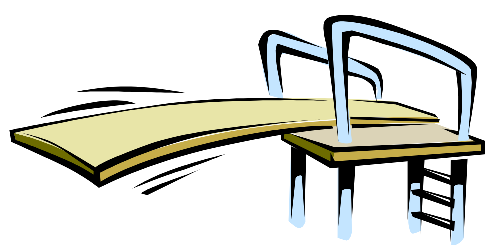
Number of Pieces of Spaghetti in the Bundle	Amount of Deflection for Each Team												Average
	A	B	C	D	E	F	G	H	I	J	K	L	
1													
2													
3													
4													
5													
6													
7													
8													


Deflection measurement grid (units)
Fold this flap back and tape to table top. Grid will hang below table.
Cut from bottom of page up to this point.
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

How Are They Alike? How Are They Different?







Fallingwater

A cantilever is a structure that is secured at only one end and is extended with a load on the other end. Diving boards and airplane wings are examples of cantilevers. One of the most famous examples of a cantilever, which is shown below, is the Frank Lloyd Wright designed home, Fallingwater. The strength of a cantilever can be affected by variables such as length, load, cross sectional area, temperature, or elasticity. In this activity, you will be investigating the relationship between the thickness of a cantilever and the deflection in the cantilever when weight is added at the end.

Making a Spaghetti Cantilever

A cantilever is a horizontal beam that is secured at only one end. Diving boards and airplane wings are examples of cantilevers. Cantilever cranes are used to lift heavy building materials and are commonly used in high rise construction and in ship building. The strength of a cantilever can be affected by variables such as length, load, cross sectional area, temperature, or elasticity. In this activity, you will be investigating the relationship between the thickness of a cantilever and the deflection in the cantilever when a certain mass is added at the end.

In this investigation, you will keep the length of a piece of spaghetti (or other type of straight pasta) that is hanging over the edge of a table constant as you collect data on how much the spaghetti deflects, that is bends, from its original horizontal position. The number of pieces of spaghetti will change as you bundle pieces together to form cantilevers of varying thickness. Since you want to keep all variables (except for the ones you are investigating) constant, make sure to pay attention to the hints listed with the instructions. In your group determine who will take on the following duties.

Materials manager: Responsible for collecting necessary materials, directing the team in setting up the investigation, and for returning all materials after the investigation

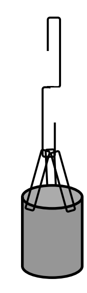
Measures manager: Responsible for setting up and reading measurements from the meter stick and ruler as the investigation proceeds

Data manager: Responsible for recording the necessary measurements in the table, sharing the data with the team, and leading the team in preparing various representations of the data, including the use of a graphing calculator

Data Collection Set-up Instructions

Step 1. The Materials manager should get the necessary materials and begin to make bundles of spaghetti. Team members should help with making the bundles. Each bundle of 1, 2, 3, 4, 5, 6, 7, and 8 pieces of spaghetti should be taped on each end and in the center. Since spaghetti is not all exactly the same length, make sure the pieces are lined up evenly on the end which will be extended over the edge of the table. *Hint: Make sure the bundles are taped securely and tightly.*

Step 2. Create a basket out of a film canister, a paper clip, and string as shown at right. Tape a piece of string to the film canister. Partially unbend the paperclip and hook it to the string. Otherwise, tape the string directly to the spaghetti. This “basket” will be looped over the end of your spaghetti cantilever.



Step 3. Tape the meter stick to the table top as shown in the diagram which follows these instructions. You may want to tape a ruler to the table top horizontally also.

Step 4. Tape a single piece of spaghetti to the top of the table with 15 centimeters hanging over the edge of a desk. Line up the end of the spaghetti with the end of the ruler. Tape the spaghetti to the table top about 3 cm from the edge of the table and near the end of the spaghetti. Place the basket (film canister) on the end of the spaghetti that is hanging over the edge of the desk 1 centimeter from the end of the spaghetti, and then lowly place pennies in the film canister until the spaghetti breaks. *Hint: You may want to support the canister from underneath as you add each penny then slowly remove your support to see if the spaghetti holds for at least 15 seconds before adding another penny.*
Use one less penny than the number required to break one piece of spaghetti as the load in your basket for the remainder of this data collection experiment.

Step 5. Replace the broken piece of spaghetti with another single piece. Tape the hanging paperclip to the end of the spaghetti 1 cm from the end and secure with tape so that it does not slide off as the spaghetti bends.

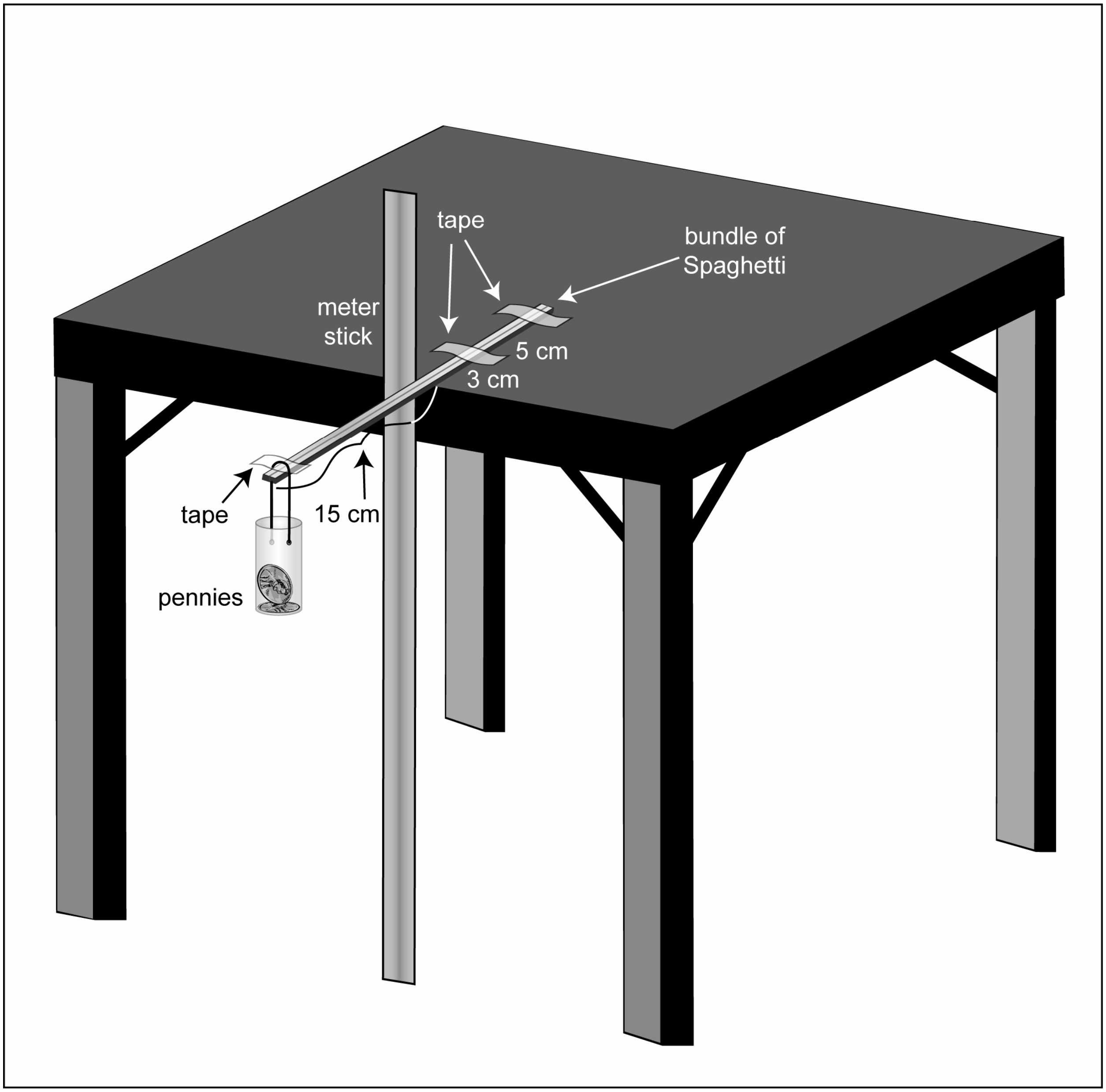
Step 6. Before adding any weight, notice the height of the spaghetti above the floor. With the pennies in the canister, carefully attach the canister basket to the hanging paperclip. *Hint: Support the canister from underneath while you are attaching it to the spaghetti then gently let go.*

Step 7. Wait 15 seconds. Measure the amount of deflection in the spaghetti by reading the number of centimeters the end of the spaghetti has dropped from the horizontal line. *Hint: To read the deflection you should kneel or squat down so your eyes are level with the end of the spaghetti. Use the eraser end of a pencil to help you measure the deflections.*
Record your measurements in the data table.

Step 8. Repeat the procedure with two pieces of spaghetti taped together and taped to the table hanging 15 centimeters over the edge. Measure the deflection of the bundle of spaghetti and record the measurement in the data table.

Step 9. Continue repeating the procedure with additional pieces of spaghetti until you have eight pieces taped together. Continue to record your data.

Spaghetti Cantilever Set Up



Spaghetti Cantilever Activity Recording Sheet

Group Name ___________________________

1. Fill in the table below with the data you collected.

Number of Pieces of Spaghetti in the Bundle (x)	Amount of Deflection in the Spaghetti (cm) (y)	Product of x and y ($x \cdot y$)
1		
2		
3		
4		
5		
6		
7		
8		

2. What are the two quantities which vary in this investigation?

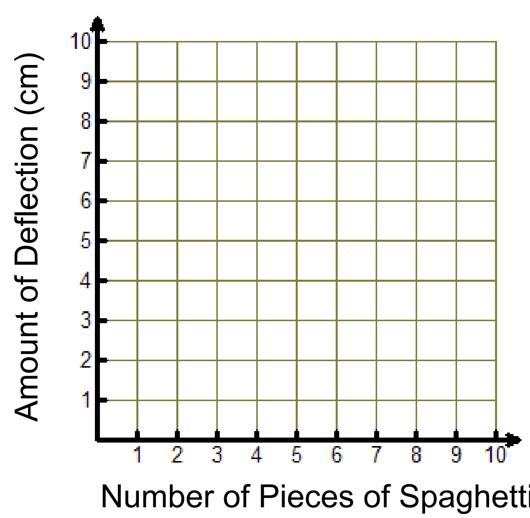
   x represents ____

   y represents ____

3. Write a dependency statement relating the two variables. Use the quantities described above, not just "*x*" and "*y*".

4. What are a reasonable domain and range for the set of data?

5. Enter the data you collected into your graphing calculator and sketch the resulting scatterplot on the grid provided.



6. What happens to the amount of deflection as the number of pieces of spaghetti increases?

7. Is this data set discrete or continuous? Why?

8. Does the set of data represent a function? Why?

9. What kind of function would best fit your data and scatterplot? Why?

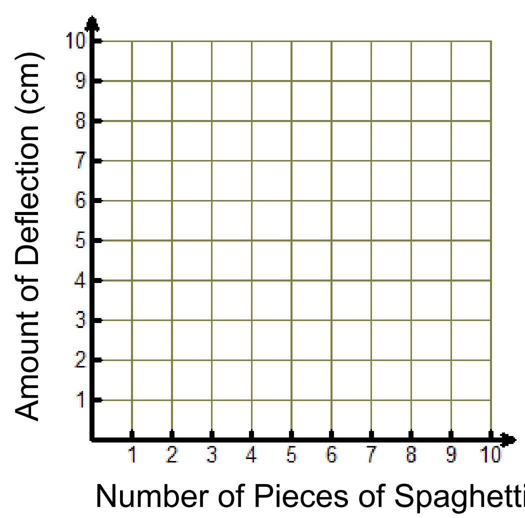
10. Use what you know about inverse relationships to write a function rule for the spaghetti cantilever data.

11. To get a better model add your set of data to the data of the entire class. Each group should send its Data manager to the overhead to fill in the data collected for your group. Record the additional data in the table below. Find the average deflection for each bundle of spaghetti for the entire class.

Number of Pieces of Spaghetti in the Bundle	**Amount of Deflection for Each Team**												**Average**
	A	B	C	D	E	F	G	H	I	J	K	L	
1													
2													
3													
4													
5													
6													
7													
8													

12. Using data from the entire class, create a scatterplot of deflection vs. number of pieces of spaghetti. How does this new scatterplot compare to the plot you made with just your group's data?

13. Write a new function rule for the class data.



Understanding the Reciprocal Function

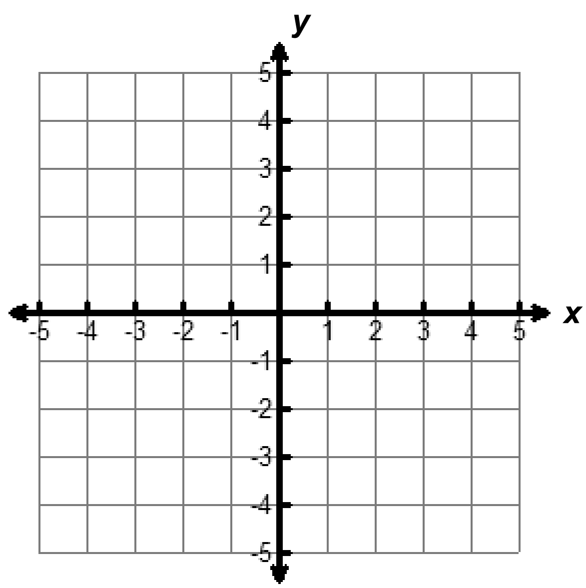
Name ___________________________

1. What is the reciprocal of the linear parent function, $f(x) = x$?
   (*Hint: The reciprocal of 2 is* $\frac{1}{2}$ *so think of the reciprocal of x.)*

2. Let's investigate some of the attributes of the function $f(x) = x$ and its reciprocal $g(x) = \frac{1}{x}$. Complete the tables below for each function. Draw a sketch of the graphs of the two functions on the same set of axes.

$f(x) = x$	
x	**y**
–3	
–2	
–1	
–0.5	
–0.1	
0	
0.1	
0.5	
1	
2	
3	

$g(x) = \frac{1}{x}$	
x	**y**
–3	
–2	
–1	
–0.5	
–0.1	
0	
0.1	
0.5	
1	
2	
3	



3. Where do the functions $f(x) = x$ and its reciprocal function $g(x) = \frac{1}{x}$ intersect? Why?

4. For any given *x* value, compare the $f(x)$ and $g(x)$ values. Write a verbal description of how these function values are related.

5. Using your graphing calculator (if necessary), fill in the tables below. Let $f(x) = x$ be Y_1, and let $g(x) = \frac{1}{x}$ be Y_2.

$Y_1 = x$		$Y_2 = \frac{1}{x}$
	Intervals where the function is increasing	
	Intervals where the function is decreasing	
	Intervals where the function is undefined	
	Coordinates of the *x*-intercepts (zeros)	
	Equations of any asymptotes	

6. What do you notice about the graphs of the linear parent function and it's reciprocal?

7. How does each function behave for values of *x* that are very close to 0?
(*Hint: Use your graphing calculator table in Ask Mode to investigate x-values between 0 and 1. Check out the negatives values between −1 and 0 also.)*

8. How does each function behave as the values of *x* get very large?
(*Hint: Use your graphing calculator table in Ask Mode to investigate x-values such as 10, 100, 1000, 10000. Check out negative values −10, −100, etc. as well.)*

9. How would you describe an asymptote to a friend?

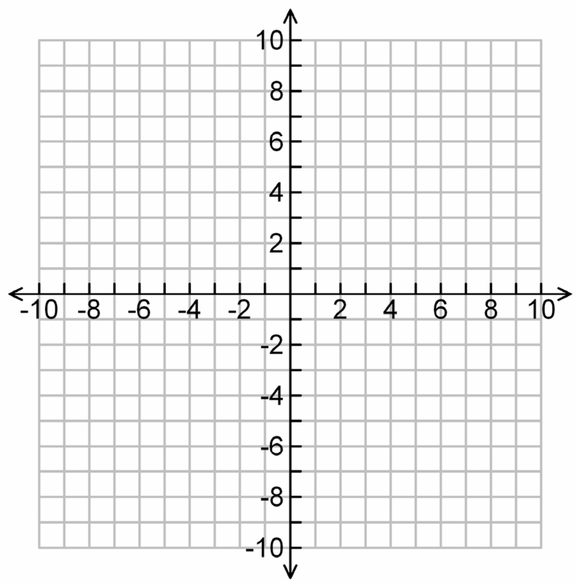
10. What are some other names for the parent function $y = \frac{1}{x}$?

Transformations to the Reciprocal Parent Function

1. Use your graphing calculator to compare the two functions.

$$Y_1 = \frac{1}{x} \text{ and } Y_2 = \frac{15}{x}$$

Sketch the graphs on the coordinate axes using two differently colored pencils.



2. Complete the table below for the given values of *x*.

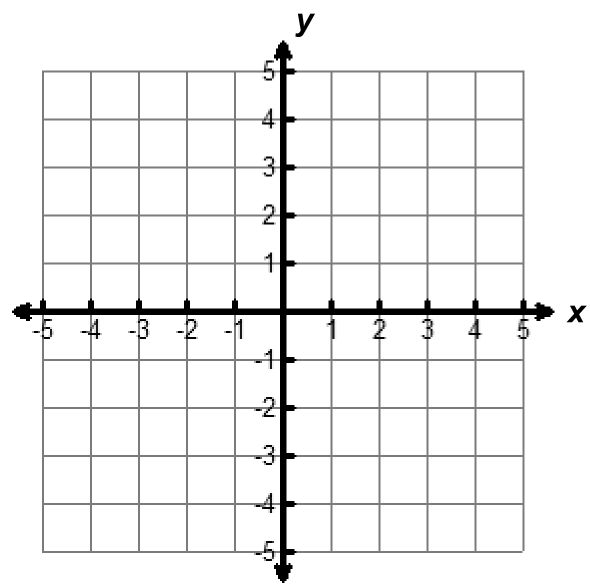
x	$\frac{1}{x}$	$\frac{15}{x}$
1		
2		
3		
5		

3. For each *x*-value in the table how does the value of $\frac{15}{x}$ compare with the value of $\frac{1}{x}$? Why?

4. The function $y = \dfrac{15}{x}$ is said to be a *transformation* of the parent reciprocal function $y = \dfrac{1}{x}$. The numerator value is 15 times greater than the numerator value in the parent function. How does this transformation affect the graph of the parent function? Why?

5. Use your graphing calculator (if necessary) to complete the tables, explore the following functions, sketch their graphs using differently colored pencils, then answer the questions that follow.

x	$Y_1 = \dfrac{1}{x}$	$Y_2 = \dfrac{3}{x}$	$Y_3 = \dfrac{0.1}{x}$	$Y_4 = -\dfrac{1}{x}$
−4				
−3				
−2				
−1				
0				
1				
2				
3				
4				



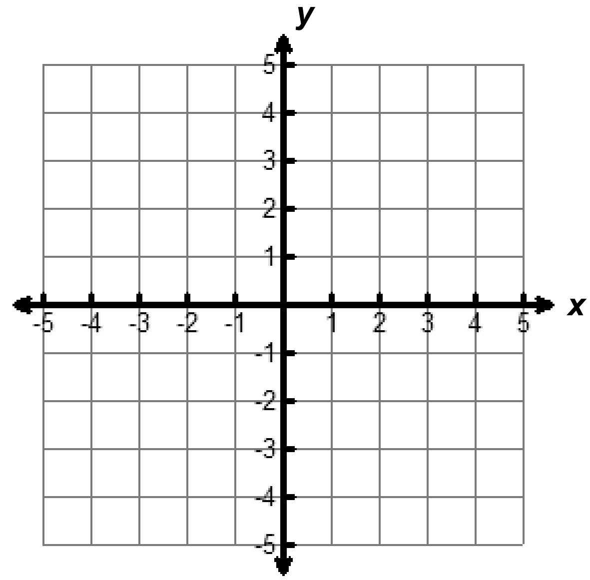
6. What happens to the reciprocal parent function, $y = \dfrac{1}{x}$, when it is multiplied by a constant greater than 1?

7. What happens to the reciprocal parent function, $y = \dfrac{1}{x}$, when it is multiplied by a constant between 0 and 1?

8. What happens to the reciprocal parent function, $y = \dfrac{1}{x}$, when it is multiplied by a negative constant?

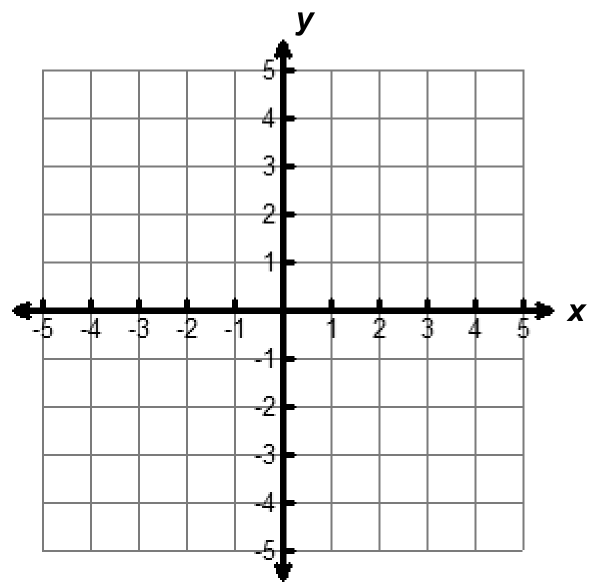
9. Use your graphing calculator (if necessary) to complete the tables, sketch the graphs of the functions from each table using differently colored pencils, then answer the questions that follow.

x	$Y_1 = \frac{1}{x}$	$Y_2 = \frac{1}{x} + 2$	$Y_3 = \frac{1}{x} - 3$
−4			
−3			
−2			
−1			
0			
1			
2			
3			
4			



10. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when a constant is added to or subtracted from the parent function? How can you see the movement in your table?

x	$Y_1 = \frac{1}{x}$	$Y_2 = \frac{1}{x+2}$	$Y_3 = \frac{1}{x-1}$
−4			
−3			
−2			
−1			
0			
1			
2			
3			
4			



11. What happens to the reciprocal parent function, $y = \frac{1}{x}$, when a constant is added to or subtracted from the *x* variable? How can you see the movement in your table?

12. Do any of the changes described above affect the asymptotes of the parent reciprocal function? Explain.

13. Do any of the changes described above affect the domain and range of the reciprocal parent function? Explain.

14. How are the transformations to the reciprocal function like the transformations to the quadratic parent function?

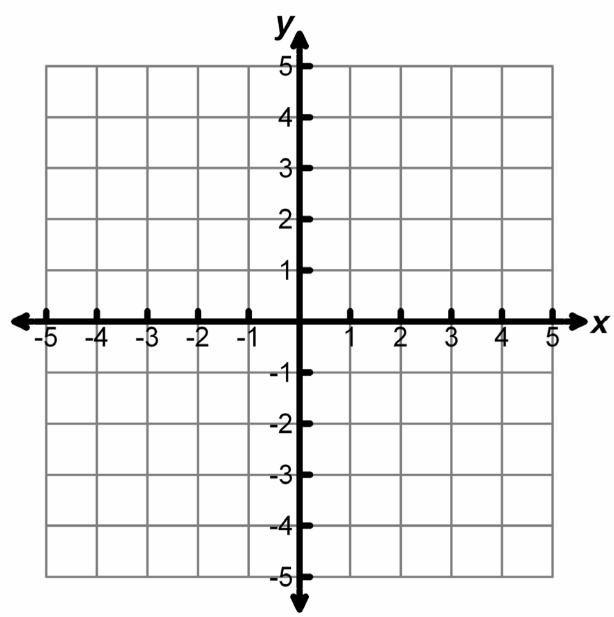
15. To summarize all of the transformations that you have studied here, look at the function below. Predict, describe, and then sketch the effects of the transformations to the parent reciprocal function. Also describe any changes to the asymptotes, domain, and range.

$$f(x) = \frac{2}{x+1} - 3$$

Asymptotes:

Domain:

Range:



Assessment Name ______________________

Let's Be Rational!

For functions 1 – 4 below, match each function to its graph and matching table of values. Write the letter of the graph and table in the grid provided.

	Function	Table *(Write the letter.)*	Graph *(Write the letter.)*
1	$y = \frac{1}{x+2}$		
2	$y = \frac{1}{x} + 2$		
3	$y = \frac{1}{x} - 2$		
4	$y = -\frac{2}{x}$		

A

x	y
−3	$\frac{2}{3}$
−2	1
−1	2
0	Und
1	−2
2	−1
3	$-\frac{2}{3}$

B

x	y
−3	-1
−2	Und
−1	1
0	$\frac{1}{2}$
1	$\frac{1}{3}$
2	$\frac{1}{4}$
3	$\frac{1}{5}$

C

x	y
−3	$1\frac{2}{3}$
−2	$1\frac{1}{2}$
−1	1
0	Und
1	3
2	$2\frac{1}{2}$
3	$2\frac{1}{3}$

D

x	y
−3	$-\frac{1}{5}$
−2	$-\frac{1}{4}$
−1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	−1
2	Und
3	1

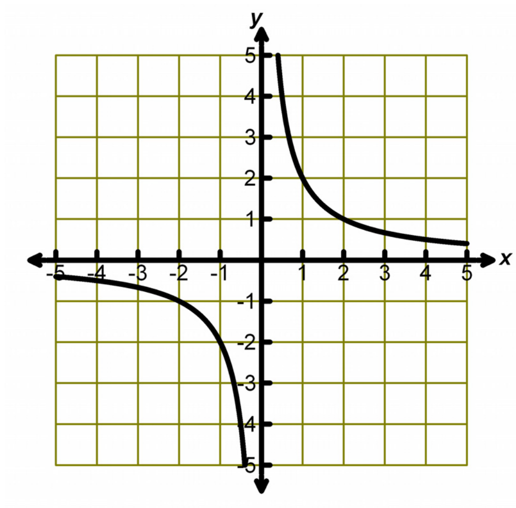
E

x	y
−3	$-\frac{2}{3}$
−2	−1
−1	−2
0	Und
1	2
2	1
3	$\frac{2}{3}$

F

x	y
−3	$-2\frac{1}{3}$
−2	$-2\frac{1}{2}$
−1	−3
0	Und
1	−1
2	$-1\frac{1}{2}$
3	$-1\frac{2}{3}$

G



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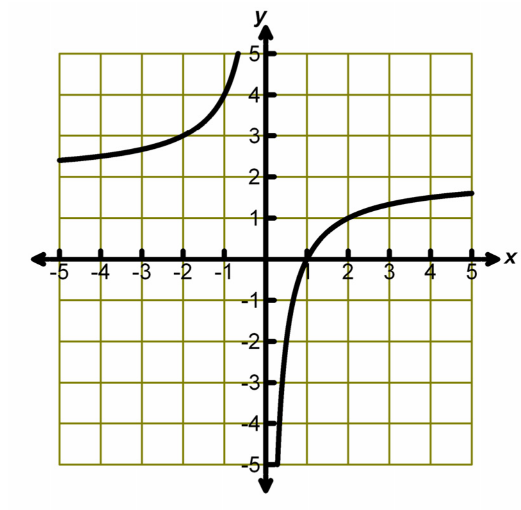

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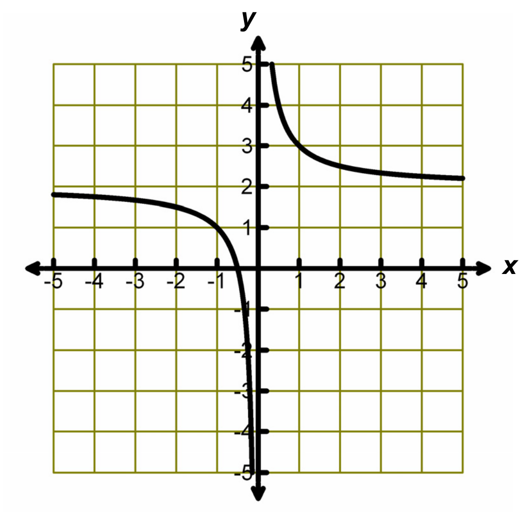

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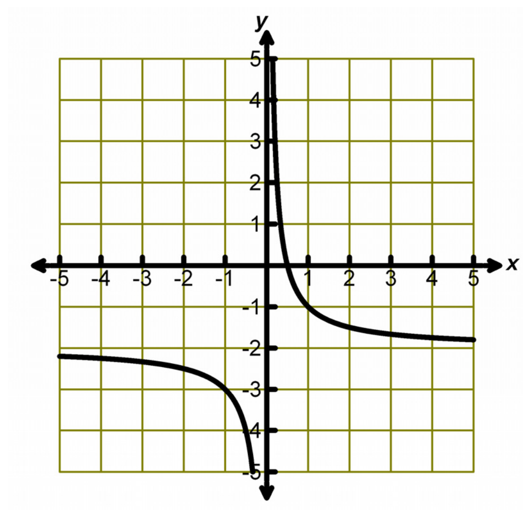

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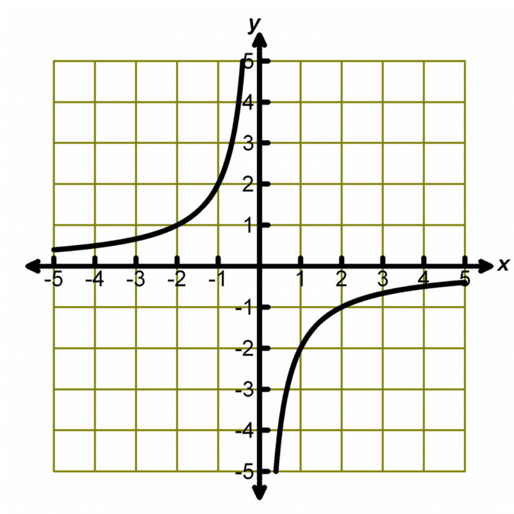

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